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Finite element formulation for cylindrical bending of a transversely compressible sandwich plate, based on assumed transverse strains

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Abstract

In order to construct a plate theory for cylindrical bending of a sandwich plate with isotropic homogeneous thick face sheets and an isotropic core, the authors make simplifying assumptions regarding distribution of transverse strain components in the thickness direction. It is assumed that the transverse strains (i.e. ε_{xz} and ε_{zz}) do not vary in the thickness direction within the face sheets and the core, but can be different functions of the in-plane coordinate in different sublaminates (the face sheets and the core). The purpose of this work is to investigate the accuracy of the theory, based on such assumptions and upon the continuity of displacements and transverse stresses at the interfaces between the layers. The finite element formulation, based on this plate theory, is made using degrees of freedom related to transverse strains and datum surface displacements. The in-plane direct stress, σ_{xx}, is computed from the constitutive equations, and the improved values of transverse stress components are computed by integration of equilibrium equations $\sigma_{ij,j} = 0$ in a post process stage. The values of the transverse strains can also be computed in the post process stage by substituting the improved transverse stresses into the constitutive relations. The improved transverse strains, unlike the assumed ones, vary in the thickness direction within a sublaminate. A problem of cylindrical bending of a simply supported plate under a uniform load on the upper surface is considered, and comparison is made between the in-plane stress, improved transverse stresses (obtained by integration of equilibrium equations) and displacements, computed from the plate theory with the corresponding values from an exact elasticity solution. This comparison showed that good agreement of both solutions is achieved. The model of a sandwich plate in cylindrical bending, presented in the present paper, has a wider range of applicability than the models presented in literature so far: it can be applied to plates with both thin and thick faces, with the cores both transversely rigid and transversely flexible, and to the plates with a wide range of thickness-to-length ratios. Published by Elsevier Science Ltd.

Keywords: Sandwich plate; Layerwise theory; Finite element formulation

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1. Introduction

Sandwich structures are used in a variety of load-bearing applications. The sandwich plates have the well pronounced zigzag variation of the in-plane displacements in the thickness direction, due to their high thickness-to-length ratios and large difference of values of elastic moduli of the face sheets and the core. Such characteristics of the sandwich plates make it necessary to use the layerwise approach in their analysis, the idea of which is to introduce the separate simplifying assumptions regarding the through-the-thickness variation of displacements, strains or stresses within each face sheet and the core. Many researchers studied the sandwich plate with thick, light-weight, vertically stiff cores and thin rigid face sheets, using discrete-layer (or layerwise) models. Most of the layerwise models of such structures are based on the piecewise linear through the thickness in-plane displacement approximations in addition to constant (though the thickness) transverse displacements (Reissner, 1948; Grigolyuk, 1957; Yu, 1959; Plantema, 1966; Allen, 1969; Kanematsu et al., 1988; Monforton and Ibrahim, 1975; Mukhopadhyay and Sierakowsky, 1990; Lee et al., 1993).

The assumption of linear variation of in-plane displacements in the thickness direction, i.e. the assumption, that the cross-sections of the core and the face sheets remain plane after deformation, holds only for the cross-sections that are far from supports or locations of concentrated and partially distributed loads. Therefore, the discrete-layer models with higher-order through-the-thickness displacement approximations for each layer (Mushtari, 1960; Chan and Foo, 1977; Gutierrez and Webber, 1980; Kutilowski and Myslecki, 1991; Liu and Chen, 1991; Paimushin, 1990) produce more accurate results.

The modern cores are usually made of plastic foams and non-metallic honeycombs, like Aramid and Nomex. These cores have properties similar to those used traditionally (for example, metallic honeycombs), but due to their transverse compressibility (i.e. ability of such cores to change height under applied loads) the direct transverse strain, ε_{zz}, becomes important. Therefore, the models of the sandwich plates with the cores made of plastic foams or non-metallic honeycombs must not exclude the change of height of the core. Frostig et al. (1992) developed a theory of a sandwich beam with thin face sheets in which account is taken of transverse compressibility of the core, and the longitudinal displacement in the core varies nonlinearly in the thickness direction. In this theory the longitudinal displacement in the face sheets varies linearly in the thickness direction, and the transverse displacement of the face sheets does not vary in the thickness direction (i.e. the transverse direct strain, ε_{zz}, in the face sheets is assumed to be equal to zero in the expression for the strain energy). The transverse shear strain, ε_{xz}, in the face sheets is also considered to be negligibly small in the expression for the strain energy, that is used for variational derivation of the differential equations for the unknown functions. The transverse shear stress in the face sheets can be computed by integration of the pointwise equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$.

Under certain circumstances, when the face sheets are thick, when the plate is loaded by a concentrated or partially distributed load, or when the plate is on an elastic foundation, taking account of the direct transverse strain, ε_{zz}, in the face sheets and the transverse shear strain, ε_{xz}, in the face sheets in the expression for the strain energy allows one to obtain a higher accuracy of the stress computation. Besides, in order to achieve a high accuracy of stress computation in the thick face sheets, a model for such a plate must assume or lead to the nonlinear through-the-thickness variation of the in-plane displacements not only in the core (as in the works of Frostig et al.), but also in the face sheets.

Construction of a computational scheme that satisfies these requirements can be approached, for example, with the help of the layerwise laminated plate theory of Reddy (1996), which is a generalization of many other displacement-based layerwise theories of laminated plates. In this theory the displacement field in the kth layer is written as $u^{(k)}(x,y,z,t) = \sum_{j=1}^{m} u_j^{(k)}(x,y,t)\phi_j^{(k)}(z)$, $v^{(k)}(x,y,z,t) = \sum_{j=1}^{m} v_j^{(k)}(x,y,t)\phi_j^{(k)}(z)$, $w^{(k)}(x,y,z,t) = \sum_{j=1}^{n} w_j^{(k)}(x,y,t)\psi_j^{(k)}(z)$, where $u_j^{(k)}(x,y,t)$, $v_j^{(k)}(x,y,t)$, $w_j^{(k)}(x,y,t)$ are the unknown functions and $\phi_j^{(k)}(z)$ and $\psi_j^{(k)}(z)$ are chosen to be the Lagrange interpolation functions of the thickness coordinate, in order to provide the required continuity of displacements and discontinuity of the transverse strains across the interface between adjacent thickness subdivisions. This theory allows one to achieve high accuracy of the transverse stress computation in the composite laminates, but for this purpose it requires a large number of thickness subdivisions of the laminate. This leads to a large number of the unknown functions and degrees of freedom in a finite element model. In effect, the finite element model, based on this generalized layerwise laminated plate theory is equivalent to the three-dimensional finite element model. In order to reduce the number of the unknown functions in the layerwise model of a laminated plate, one can use the concept of a sublaminate, i.e. make the number of thickness subdivisions less than the number of material layers, and deal with the material properties, averaged through the thickness of a sublaminate. In a model of the sandwich plate it is natural to choose three sublaminates: the two face sheets and the core. With such a small number of the sublaminates, the nature of assumptions on the through-the-thickness variation of displacements can have a large effect on the accuracy of the computed stresses. Besides, the actual through-the-thickness variation of displacements can depend on the character of applied loads and boundary conditions. Therefore, in a layerwise model of the sandwich plate with only three sublaminates, it is desirable to have a flexibility in the choice of the functions that represent through-the-thickness variation of displacements. Of course, the Lagrange interpolation polynomials, that represent the thickness variation of the displacements within a sublaminate in the Reddy's layerwise theory, can be chosen to be of any desired degree, but such increase of the degree of the Lagrange interpolation polynomials leads to the increase of the number of the unknown functions.

In the present paper, we construct a computational scheme for analysis of the sandwich plate, in which the simplifying assumptions that lead to a plate-type theory are made with respect to the variation of the transverse strains in the thickness direction of the face sheets and the core of the sandwich plates. The displacements are then obtained by integration of these assumed transverse strains, and the constants of integration are chosen to satisfy the conditions of continuity of the displacements across the borders between the face sheets and the core. In such a method, the required continuity of the displacements in the thickness direction is satisfied regardless of the assumed type of through-the-thickness distribution of the transverse strains. This leads to a larger number of choices of simplifying assumptions about the variation of strains (and, therefore, displacements) in the thickness direction, and, therefore allows a better adjustment of the computational scheme to the conditions under which the sandwich plate is analyzed by a layerwise method with only three sublaminates (being the face sheets and the core). The transverse stresses are computed by integration of the pointwise equilibrium equations, $\sigma_{ij,j} = 0$, that leads to satisfaction of conditions of continuity of the transverse stresses across the boundaries between the face sheets and the core and satisfaction of stress boundary conditions on the upper and lower surfaces of the plate.

In the present paper, we study the accuracy of the model based on the simplest of such assumptions that do not ignore in the expression for the strain energy the transverse shear and normal strains in the face sheets. We assume that the transverse strains do not vary in the thickness direction within the face sheets and the core, but can be different functions of the in-plane coordinate in the face sheets and the core. In the post-process stage these first approximations of the transverse strains can be improved by substituting the transverse stresses, obtained by integration of the pointwise equilibrium equations, $\sigma_{ij,j} = 0$, into the strain–stress relations. The improved values of the transverse strains depend on the z-coordinate. In this model, the transverse displacement, obtained by integration of the assumed transverse normal strain, varies linearly in the thickness direction within a sublaminate, and the in-plane displacement obtained by integration of the assumed transverse shear strains varies quadratically within the thickness of a sublaminate.

The theory of the sandwich plate, presented in this paper, does not require so many degrees of freedom in the finite element formulation as the generalized laminated plate theory of Reddy and has the wider range of applicability than the other theories, discussed above. It can be used for the analysis of sandwich plates with large and small thickness-to-length ratios, with thick and thin face sheets, with transversely rigid and transversely flexible face sheets and cores. Besides, in the finite element analysis of sandwich plates with small thickness-to-length ratios, the shear locking phenomenon does not occur.

2. Formulation of the Problem

Let us consider cylindrical bending of a wide sandwich plate with the isotropic face sheets and core (Fig. 1a and b). The upper and lower surfaces of the plate are under loads with intensities (force per unit length) q_{u} (upper surface intensity) and q_{l} (lower surface intensity). By q_{u} and q_{l} we denote not absolute values of the load intensities, but projections of the load intensities on the z-axis, i.e. q_{u} and q_{l} can be positive or negative, depending on direction of the load.

In the following equations, the superscript k in parentheses denotes a number of a sublaminate: $k = 1$ means the lower face sheet, $k = 2$ – the core and $k = 3$ – the upper face sheet.

The equations of linear elasticity for the kth sublaminate ($k = 1, 2, 3$), as applied to this problem, have the form

Equilibrium equations :

$$\sigma^{(k)}_{xx,x} + \sigma^{(k)}_{xz,z} = 0, \tag{1}$$

$$\sigma^{(k)}_{xz,x} + \sigma^{(k)}_{zz,z} = 0, \tag{2}$$

strain–displacement relations for plane strain

$$\varepsilon^{(k)}_{xx} = u^{(k)}_{,x}, \tag{3}$$

$$\varepsilon^{(k)}_{zz} = w^{(k)}_{,z}, \tag{4}$$

$$\varepsilon^{(k)}_{xz} = \tfrac{1}{2}\left(u^{(k)}_{,z} + w^{(k)}_{,x}\right), \tag{5}$$

$$\varepsilon^{(k)}_{yy} = \varepsilon^{(k)}_{yz} = \varepsilon^{(k)}_{xy} = 0. \tag{6}$$

The constitutive relations for plane strain are

$$\sigma^{(k)}_{xx} = \frac{E^{(k)}}{(1+v^{(k)})(1-2v^{(k)})}\left[(1-v^{(k)})\varepsilon^{(k)}_{xx} + v\varepsilon^{(k)}_{zz}\right], \tag{7}$$

$$\sigma^{(k)}_{zz} = \frac{E^{(k)}}{(1+v)(1-2v)}\left[(1-v)\varepsilon^{(k)}_{zz} + v\varepsilon^{(k)}_{xx}\right], \tag{8}$$

$$\sigma^{(k)}_{yy} = \frac{E^{(k)}v^{(k)}}{(1+v^{(k)})(1-2v^{(k)})}\left(\varepsilon^{(k)}_{xx} + \varepsilon^{(k)}_{zz}\right) = v^{(k)}\left(\sigma^{(k)}_{xx} + \sigma^{(k)}_{zz}\right), \tag{9}$$

$$\sigma^{(k)}_{xz} = \frac{E^{(k)}}{(1+v^{(k)})}\varepsilon^{(k)}_{xz}, \tag{10}$$

$$\sigma^{(k)}_{xy} = \sigma^{(k)}_{yz} = 0, \tag{11}$$

or, in the inverse form,

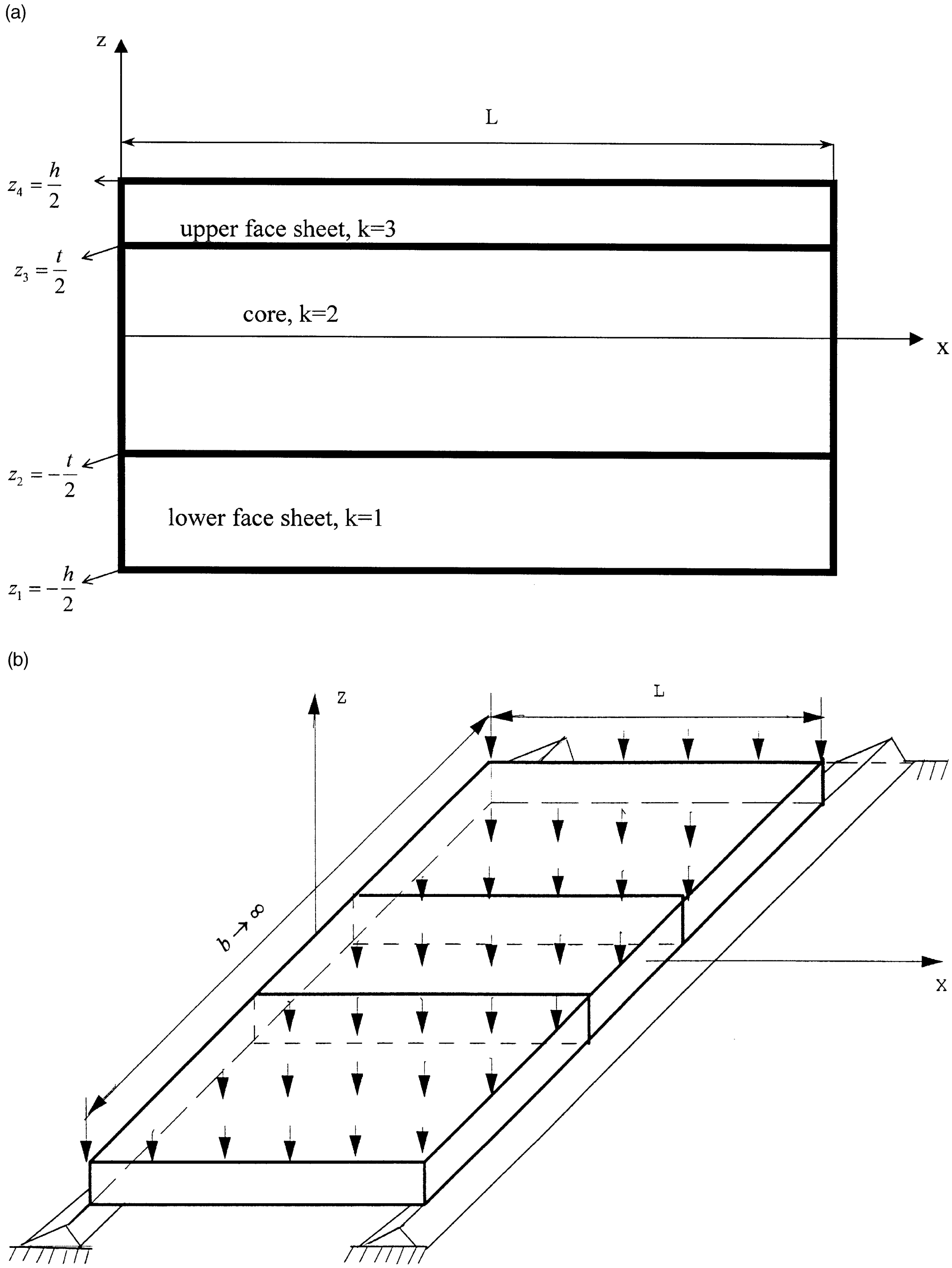

Fig. 1. (a) The coordinate system and notations for sandwich plate. (b) Wide simply supported plate in cylindrical bending under a uniform load on the upper surface.

$$\varepsilon_{xx}^{(k)} = \frac{1-\left(\nu^{(k)}\right)^2}{E^{(k)}}\left(\sigma_{xx}^{(k)} - \frac{\nu^{(k)}}{1-\nu^{(k)}}\sigma_{zz}^{(k)}\right), \tag{12}$$

$$\varepsilon_{zz}^{(k)} = \frac{1-\left(\nu^{(k)}\right)^2}{E^{(k)}}\left(\sigma_{zz}^{(k)} - \frac{\nu^{(k)}}{1-\nu^{(k)}}\sigma_{xx}^{(k)}\right), \tag{13}$$

$$\varepsilon_{xz}^{(k)} = \frac{1+\nu^{(k)}}{E^{(k)}}\sigma_{xz}^{(k)}, \tag{14}$$

$$\varepsilon_{yy}^{(k)} = \varepsilon_{xy}^{(k)} = \varepsilon_{yz}^{(k)} = 0, \tag{15}$$

where $\sigma_{ij}^{(k)}$ $(i,j = x,y,z)$ are components of the stress tensor, $\varepsilon_{ij}^{(k)}$ $(i,j = x,y,z)$ are components of the strain tensor, $E^{(k)}$, the Young's modulus, $\nu^{(k)}$, the Poisson ratio.

Boundary conditions at the lower and upper surfaces are

$$\sigma_{xz}^{(1)} = 0, \quad \sigma_{zz}^{(1)} = -\frac{q_l}{b} \quad \text{at } z = -\frac{h}{2} = z_1, \tag{16}$$

$$\sigma_{xz}^{(3)} = 0, \quad \sigma_{zz}^{(3)} = \frac{q_u}{b} \quad \text{at } z = \frac{h}{2} = z_4. \tag{17}$$

The continuity of displacements and transverse stresses at the interfaces between the core and the face sheets can be stated as

$$u^{(1)} = u^{(2)}, \quad w^{(1)} = w^{(2)}, \quad \sigma_{xz}^{(1)} = \sigma_{xz}^{(2)}, \quad \sigma_{zz}^{(1)} = \sigma_{zz}^{(2)} \quad \text{at } z = -\frac{t}{2} = z_2, \tag{18}$$

$$u^{(2)} = u^{(3)}, \quad w^{(2)} = w^{(3)}, \quad \sigma_{xz}^{(2)} = \sigma_{xz}^{(3)}, \quad \sigma_{zz}^{(2)} = \sigma_{zz}^{(3)} \quad \text{at } z = \frac{t}{2} = z_3. \tag{19}$$

Finally, the condition of static equilibrium, common for all end boundary conditions is

$$b\int_{-h/2}^{h/2}\left(\sigma_{xz}|_{x=L} - \sigma_{xz}|_{x=0}\right)\mathrm{d}z = -\int_0^L (q_l + q_u)\,\mathrm{d}x,$$

or

$$\int_{-h/2}^{-t/2}\left(\sigma_{xz}^{(1)}\Big|_0^L\right)\mathrm{d}z + \int_{-t/2}^{t/2}\left(\sigma_{xz}^{(2)}\Big|_0^L\right)\mathrm{d}z + \int_{t/2}^{h/2}\left(\sigma_{xz}^{(3)}\Big|_0^L\right)\mathrm{d}z = -\frac{1}{b}\int_0^L (q_l + q_u)\,\mathrm{d}x, \tag{20}$$

where b denotes the width of the plate. Eq. (20) means that the sum of normal forces, applied to the upper and lower surfaces, is equal to the sum of shear forces, applied at the plate's ends.

The formulation of the problem includes also the additional boundary conditions at $x = 0, L$. For example, for a plate, simply supported along the edges $x = 0, L$, the boundary conditions have the form, for mitigated (integral) stress boundary conditions, that can also be looked upon as conditions of static equilibrium,

$$\begin{aligned}
&\int_{-h/2}^{-t/2} \sigma_{xx}^{(1)}\,\mathrm{d}z = 0 \quad \text{at } x = 0, L,\\
&\int_{-t/2}^{t/2} \sigma_{xx}^{(2)}\,\mathrm{d}z = 0 \quad \text{at } x = 0, L,\\
&\int_{t/2}^{h/2} \sigma_{xx}^{(3)}\,\mathrm{d}z = 0 \quad \text{at } x = 0, L,
\end{aligned} \tag{21}$$

where t is thickness of the core, h, whole thickness of the sandwich plate

$$\int_{-h/2}^{h/2} \sigma_{xx} z\,\mathrm{d}z = 0 \quad \text{at } x = 0, L, \quad \text{or} \quad \int_{-h/2}^{-t/2} \sigma_{xx}^{(1)} z\,\mathrm{d}z + \int_{-t/2}^{t/2} \sigma_{xx}^{(2)} z\,\mathrm{d}z + \int_{t/2}^{h/2} \sigma_{xx}^{(3)} z\,\mathrm{d}z = 0 \quad \text{at } x = 0, L, \tag{22}$$

where Eq. (21) mean that the resulting horizontal forces at the ends are equal to zero, and Eq. (22) mean that the resulting moments at plate's ends are equal to zero;

Displacement boundary conditions :

$$w = 0 \quad \text{at } x = 0, L \quad \text{and} \quad z = 0. \tag{23}$$

If the boundary conditions and the load are symmetric with respect to the plane $x = L/2$, then we also have a symmetry condition

$$u\left(\frac{L}{2}\right) = 0. \tag{24}$$

3. Assumptions of the plate theory

In order to construct this plate theory, we make an assumption that initially the transverse strains do not vary in the thickness direction within a layer (a face sheet or a core) of a sandwich plate, but can be different in different layers:

$$\varepsilon_{xz}^{(k)} = \varepsilon_{xz}^{(k)}(x), \qquad \varepsilon_{zz}^{(k)} = \varepsilon_{zz}^{(k)}(x) \quad (k = 1, 2, 3). \tag{25}$$

This is the first form of the transverse strains. The second form of the transverse strains can be obtained by substituting into the strain–stress relations the transverse stresses, obtained by integration of pointwise equilibrium equations $\sigma_{ij,j} = 0$. To indicate that the assumed strains (25) are the first forms of the strains, we will also use another notation:

$$\varepsilon_{xz}^{(k)} \equiv \left(\varepsilon_{xz}^{(k)}\right)^{(\mathrm{I})}, \quad \varepsilon_{zz}^{(k)} \equiv \left(\varepsilon_{zz}^{(k)}\right)^{(\mathrm{I})}. \tag{26}$$

The notation (26), with the second upper superscript, will be used only when it is necessary to distinguish between the first and the second forms of transverse strains.

The unknown functions of the problem are

$$\begin{aligned}
&u_0(x) \equiv u^{(2)}\big|_{z=0} \equiv u\big|_{z=0}, \qquad w_0(x) \equiv w^{(2)}\big|_{z=0} \equiv w\big|_{z=0},\\
&\varepsilon_{xz}^{(1)}(x), \quad \varepsilon_{zz}^{(1)}(x), \quad \varepsilon_{xz}^{(2)}(x), \quad \varepsilon_{zz}^{(2)}(x), \quad \varepsilon_{xz}^{(3)}(x), \quad \varepsilon_{zz}^{(3)}(x).
\end{aligned} \tag{27}$$

So, there are eight unknown functions in this approach for analysis of cylindrical bending of the sandwich plate.

4. Expressions for displacements $u(x,z)$, $w(x,z)$ in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ $(k=1,2,3)$

Let us first integrate the strain–displacement relations $\varepsilon_{zz}^{(k)} = w_{,z}^{(k)}$ (Eq. (4)). For the core of the sandwich plate ($k=2$), which contains plane $z=0$, we receive

$$w^{(2)}(x,z) - \underbrace{w^{(2)}\big|_{z=0}}_{w_0(x)} = \int_0^z \frac{\partial w^{(2)}}{\partial z}\,\mathrm{d}z = \int_0^z \varepsilon_{zz}^{(2)}(x,z)\,\mathrm{d}z \quad (z_2 \leqslant z \leqslant z_3),$$

or

$$w^{(2)}(x,z) = w_0(x) + \int_0^z \varepsilon_{zz}^{(2)}(x)\,\mathrm{d}z \quad (z_2 \leqslant z \leqslant z_3) \tag{28}$$

From Eq. (28), it follows that:

$$w^{(2)}\big|_{z=z_2} = w_0 + \int_0^{z_2} \varepsilon_{zz}^{(2)}\,\mathrm{d}z. \tag{29}$$

The integration of equation $\varepsilon_{zz}^{(1)} = \partial w^{(1)}/\partial z$ from z_2 to z, where z belongs to the region of the lower face sheet ($z_1 \leqslant z \leqslant z_2$), yields

$$w^{(1)} - w^{(1)}\big|_{z=z_2} = \int_{z_2}^z \frac{\partial w^{(1)}}{\partial z}\,\mathrm{d}z = \int_{z_2}^z \varepsilon_{zz}^{(1)}\,\mathrm{d}z \quad (z_1 \leqslant z \leqslant z_2), \tag{30}$$

or, due to continuity condition $w^{(1)}\big|_{z=z_2} = w^{(2)}\big|_{z=z_2}$

$$w^{(1)} = w^{(2)}\big|_{z=z_2} + \int_{z_2}^z \varepsilon_{zz}^{(1)}\,\mathrm{d}z. \tag{31}$$

If we substitute Eq. (29) for $w^{(2)}\big|_{z=z_2}$ into Eq. (31), we receive

$$w^{(1)} = w_0 + \int_0^{z_2} \varepsilon_{zz}^{(2)}\,\mathrm{d}z + \int_{z_2}^z \varepsilon_{zz}^{(1)}\,\mathrm{d}z \quad (z_1 \leqslant z \leqslant z_2). \tag{32}$$

Analogously, if we integrate equation $\varepsilon_{zz}^{(3)} = \partial w^{(3)}/\partial z$ and satisfy the continuity condition at the interface between the second and the third zone, $w^{(3)}\big|_{z=z_3} = w^{(2)}\big|_{z=z_3}$, we receive

$$w^{(3)} = w_0 + \int_0^{z_3} \varepsilon_{zz}^{(2)}\,\mathrm{d}z + \int_{z_3}^z \varepsilon_{zz}^{(3)}\,\mathrm{d}z \quad (z_3 \leqslant z \leqslant z_4). \tag{33}$$

Integration in Eqs. (28), (32) and (33) yields

$$w^{(2)} = w_0 + \varepsilon_{zz}^{(2)} z \quad (z_2 \leqslant z \leqslant z_3), \tag{34}$$

$$w^{(1)} = w_0 + \varepsilon_{zz}^{(2)} z_2 + \varepsilon_{zz}^{(1)}(z - z_2) \quad (z_1 \leqslant z \leqslant z_2), \tag{35}$$

$$w^{(3)} = w_0 + \varepsilon_{zz}^{(2)} z_3 + \varepsilon_{zz}^{(3)}(z - z_3) \quad (z_3 \leqslant z \leqslant z_4). \tag{36}$$

Now, let us find expressions for displacements $u^{(1)}, u^{(2)}, u^{(3)}$ in terms of the unknown functions. From the strain–displacement equation (5), we receive

$$u_{,z}^{(k)} = 2\varepsilon_{xz}^{(k)} - w_{,x}^{(k)}. \tag{37}$$

Eq. (37) is integrated for each layer to yield

$$u^{(2)}(x,z) - \underbrace{u^{(2)}\big|_{z=0}}_{u_0(x)} = \int_0^z \frac{\partial u^{(2)}}{\partial z}\,\mathrm{d}z = \int_0^z \left(2\varepsilon_{xz}^{(2)} - w_{,x}^{(2)}\right)\mathrm{d}z \quad (z_2 \leqslant z \leqslant z_3), \tag{38}$$

$$u^{(1)}(x,z) - u^{(1)}\big|_{z=z_2} = \int_{z_2}^z \frac{\partial u^{(1)}}{\partial z}\,\mathrm{d}z = \int_{z_2}^z \left(2\varepsilon_{xz}^{(1)} - w_{,x}^{(1)}\right)\mathrm{d}z \quad (z_1 \leqslant z \leqslant z_2), \tag{39}$$

$$u^{(3)}(x,z) - u^{(3)}\big|_{z=z_3} = \int_{z_3}^z \frac{\partial u^{(1)}}{\partial z}\,\mathrm{d}z = \int_{z_3}^z \left(2\varepsilon_{xz}^{(1)} - w_{,x}^{(1)}\right)\mathrm{d}z \quad (z_3 \leqslant z \leqslant z_4). \tag{40}$$

When we substitute Eqs. (34)–(36) for $w^{(1)}$, $w^{(2)}$, $w^{(3)}$ into Eqs. (38)–(40), perform the integration in the resulting expressions and find the constants of integration from the conditions of continuity of displacements, u, at the interfaces between the zones,

$$u^{(1)}\big|_{z=z_2} = u^{(2)}\big|_{z=z_2}, \qquad u^{(2)}\big|_{z=z_3} = u^{(3)}\big|_{z=z_3},$$

we receive expressions for displacements $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ in terms of the unknown functions $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$

$$u^{(1)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right)z_2 - \tfrac{1}{2}\varepsilon_{zz,x}^{(2)}z_2^2 + \left(2\varepsilon_{xz}^{(1)} - w_{0,x} - \varepsilon_{zz,x}^{(2)}z_2\right)(z - z_2) - \tfrac{1}{2}\varepsilon_{zz,x}^{(1)}(z - z_2)^2 \quad (z_1 \leqslant z \leqslant z_2), \tag{41}$$

$$u^{(2)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right)z - \tfrac{1}{2}\varepsilon_{zz,x}^{(2)}z^2 \quad (z_2 \leqslant z \leqslant z_3), \tag{42}$$

$$u^{(3)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right)z_3 - \tfrac{1}{2}\varepsilon_{zz,x}^{(2)}z_3^2 + \left(2\varepsilon_{xz}^{(3)} - w_{0,x} - \varepsilon_{zz,x}^{(2)}z_3\right)(z - z_3) - \tfrac{1}{2}\varepsilon_{zz,x}^{(3)}(z - z_3)^2 \quad (z_3 \leqslant z \leqslant z_4). \tag{43}$$

5. In-plane strains $\varepsilon_{xx}^{(1)}$, $\varepsilon_{xx}^{(2)}$, $\varepsilon_{xx}^{(3)}$ in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$

The result of substitution of Eqs. (41)–(43) into the strain–displacement relations $\varepsilon_{xx}^{(k)} = \partial u^{(k)}/\partial x$ can be written as

$$\varepsilon_{xx}^{(1)} = \varphi_{xx0}^{(1)} + \varphi_{xx1}^{(1)}z + \varphi_{xx2}^{(1)}z^2, \tag{44}$$

$$\varepsilon_{xx}^{(2)} = \varphi_{xx0}^{(2)} + \varphi_{xx1}^{(2)}z + \varphi_{xx2}^{(2)}z^2, \tag{45}$$

$$\varepsilon_{xx}^{(3)} = \varphi_{xx0}^{(3)} + \varphi_{xx1}^{(3)}z + \varphi_{xx2}^{(3)}z^2, \tag{46}$$

where

$$\varphi_{xx0}^{(1)} = u_{0,x} + 2z_2\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(1)}\right) + \tfrac{1}{2}z_2^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(1)}\right), \tag{47}$$

$$\varphi_{xx1}^{(1)} = 2\varepsilon_{xz,x}^{(1)} - w_{0,xx} + z_2\left(\varepsilon_{zz,xx}^{(1)} - \varepsilon_{zz,xx}^{(2)}\right), \tag{48}$$

$$\varphi_{xx2}^{(1)} = -\tfrac{1}{2}\varepsilon_{zz,xx}^{(1)}, \tag{49}$$

$$\varphi_{xx0}^{(2)} = u_{0,x}, \tag{50}$$

$$\varphi_{xx1}^{(2)} = 2\varepsilon_{xz,x}^{(2)} - w_{0,xx}, \tag{51}$$

$$\varphi_{xx2}^{(2)} = -\tfrac{1}{2}\varepsilon_{zz,xx}^{(2)}, \tag{52}$$

$$\varphi_{xx0}^{(3)} = u_{0,x} + 2z_3\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(3)}\right) + \tfrac{1}{2}z_3^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(3)}\right), \tag{53}$$

$$\varphi_{xx1}^{(3)} = 2\varepsilon_{xz,x}^{(3)} - w_{0,xx} + z_3\left(\varepsilon_{zz,xx}^{(3)} - \varepsilon_{zz,xx}^{(2)}\right), \tag{54}$$

$$\varphi_{xx2}^{(3)} = -\tfrac{1}{2}\varepsilon_{zz,xx}^{(3)}. \tag{55}$$

Using the found expressions for the in-plane strains in terms of the unknown functions, we can write the following matrix relations, which will be useful in writing an expression for strain energy in terms of the unknown functions:

$$\underset{(3\times1)}{\left\{\varepsilon^{(k)}(x,z)\right\}} = \underset{(3\times5)}{[Z(z)]}\underset{(5\times1)}{\left\{f^{(k)}(x)\right\}} \quad (k = 1,2,3), \tag{56}$$

where

$$\left\{\varepsilon^{(k)}(x,z)\right\} = \begin{Bmatrix} \varepsilon_{xx}^{(k)} \\ 2\varepsilon_{xz}^{(k)} \\ \varepsilon_{zz}^{(k)} \end{Bmatrix}, \tag{57}$$

$$[Z(z)] = \begin{bmatrix} 1 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{58}$$

$$\left\{f^{(k)}(x)\right\} = \begin{Bmatrix} \varphi_{xx0}^{(k)} \\ \varphi_{xx1}^{(k)} \\ \varphi_{xx2}^{(k)} \\ 2\varepsilon_{xz}^{(k)} \\ \varepsilon_{zz}^{(k)} \end{Bmatrix}. \tag{59}$$

6. Expressions for in-plane stresses and the first forms of transverse stresses in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ $(k = 1,2,3)$

We will distinguish between the two forms of expressions for the transverse stresses in terms of the unknown functions: the first form, ${}^{\mathrm{H}}\sigma_{xz}^{(k)} \equiv^{(\mathrm{I})} \sigma_{xz}^{(k)}$ and ${}^{\mathrm{H}}\sigma_{zz}^{(k)} \equiv^{(\mathrm{I})} \sigma_{xz}^{(k)}$, obtained from the Hooke's law by substituting into the stress–strain relations the assumed transverse strains from Eq. (25), (which we also called the first form of the transverse strains and denoted as $\varepsilon_{xz}^{(k)} \equiv \left(\varepsilon_{xz}^{(k)}\right)^{(\mathrm{I})}$, $\varepsilon_{zz}^{(k)} \equiv \left(\varepsilon_{zz}^{(k)}\right)^{(\mathrm{I})}$), and the second form of transverse stresses, obtained from the equilibrium Eqs. (1) and (2), which will be denoted as $\sigma_{xz}^{(k)} \equiv {}^{(\mathrm{II})}\sigma_{xz}^{(k)}$ and $\sigma_{zz}^{(k)} \equiv {}^{(\mathrm{II})}\sigma_{zz}^{(k)}$. The second form of expressions for the transverse stresses satisfy the boundary conditions, stated in Eqs. (16) and (17), at the upper and lower surfaces of the plate, and the conditions of continuity of the transverse stresses at the interfaces between the layers with different material properties

(Eqs. (18) and (19)). The first form of the transverse stresses cannot satisfy the mentioned boundary and continuity conditions. Therefore, the second form of the transverse stresses is more accurate than the first form. The expressions for the in-plane stresses $\sigma_{xx}^{(k)}$ in terms of the unknown functions will be determined only from the Hooke's law and, therefore, these expressions will be denoted by ${}^{\mathrm{H}}\sigma_{xx}^{(k)}$. There will only be one form of expressions for the in-plane stresses in terms of the unknown functions. Constitutive equations. (7), (8) and (10) can be written in matrix form as follows:

$$\underset{(3\times1)}{\left\{{}^{\mathrm{H}}\sigma^{(k)}\right\}} = \underset{(3\times3)}{\left[C^{(k)}\right]}\underset{(3\times1)}{\left\{\varepsilon^{(k)}\right\}} \quad (k=1,2,3), \tag{60}$$

where

$$\left\{{}^{\mathrm{H}}\sigma^{(k)}\right\} = \begin{Bmatrix} {}^{\mathrm{H}}\sigma_{xx}^{(k)} \\ {}^{\mathrm{H}}\sigma_{xz}^{(k)} \\ {}^{\mathrm{H}}\sigma_{zz}^{(k)} \end{Bmatrix}, \tag{61}$$

$$\left[C^{(k)}\right] = \frac{E^{(k)}}{1+v}\begin{bmatrix} \frac{1-v}{1-2v} & 0 & \frac{v}{1-2v} \\ 0 & \frac{1}{2} & 0 \\ \frac{v}{1-2v} & 0 & \frac{1-v}{1-2v} \end{bmatrix}, \tag{62}$$

$$\left\{\varepsilon^{(k)}\right\} = \begin{Bmatrix} \varepsilon_{xx}^{(k)} \\ 2\varepsilon_{xz}^{(k)} \\ \varepsilon_{zz}^{(k)} \end{Bmatrix}. \tag{63}$$

Eq. (56) can then be used, and we can write

$$\begin{Bmatrix} {}^{\mathrm{H}}\sigma_{xx}^{(k)} \\ {}^{\mathrm{H}}\sigma_{xz}^{(k)} \\ {}^{\mathrm{H}}\sigma_{zz}^{(k)} \end{Bmatrix} = \underset{(3\times3)}{\left[C^{(k)}\right]}\underset{(3\times5)}{[Z(z)]}\underset{(5\times1)}{\left\{f^{(k)}(x)\right\}}. \tag{64a}$$

Next, in order to develop a finite element formulation, we have to write the virtual work principle $\delta U - \delta' W = 0$ in terms of the variations of the unknown functions. Here by U we denote the strain energy, and by $\delta' W$ – virtual work of the external (generally nonconservative) forces.

7. Strain energy of the sandwich plate

Strain energy of the sandwich plate consists of strain energies of the face sheets and the core. Therefore, it can be written as follows:

$$\begin{aligned} U = {} & \frac{1}{2}\iiint_{(V_1)}\left({}^{\mathrm{H}}\sigma_{xx}^{(1)}\varepsilon_{xx}^{(1)} + 2{}^{\mathrm{H}}\sigma_{xz}^{(1)}\varepsilon_{xz}^{(1)} + {}^{\mathrm{H}}\sigma_{zz}^{(1)}\varepsilon_{zz}^{(1)} + {}^{\mathrm{H}}\sigma_{yy}^{(1)}\underbrace{\varepsilon_{yy}^{(1)}}_{0} + 2\underbrace{{}^{\mathrm{H}}\sigma_{xy}^{(1)}}_{0}\underbrace{\varepsilon_{xy}^{(1)}}_{0} + 2\underbrace{{}^{\mathrm{H}}\sigma_{yz}^{(1)}}_{0}\underbrace{\varepsilon_{yz}^{(1)}}_{0}\right)\mathrm{d}V \\ & + \frac{1}{2}\iiint_{(V_2)}\left({}^{\mathrm{H}}\sigma_{xx}^{(2)}\varepsilon_{xx}^{(2)} + 2{}^{\mathrm{H}}\sigma_{xz}^{(2)}\varepsilon_{xz}^{(2)} + {}^{\mathrm{H}}\sigma_{zz}^{(2)}\varepsilon_{zz}^{(2)} + {}^{\mathrm{H}}\sigma_{yy}^{(2)}\underbrace{\varepsilon_{yy}^{(2)}}_{0} + 2\underbrace{{}^{\mathrm{H}}\sigma_{xy}^{(2)}}_{0}\underbrace{\varepsilon_{xy}^{(2)}}_{0} + 2\underbrace{{}^{\mathrm{H}}\sigma_{yz}^{(2)}}_{0}\underbrace{\varepsilon_{yz}^{(2)}}_{0}\right)\mathrm{d}V \\ & + \frac{1}{2}\iiint_{(V_3)}\left({}^{\mathrm{H}}\sigma_{xx}^{(3)}\varepsilon_{xx}^{(3)} + 2{}^{\mathrm{H}}\sigma_{xz}^{(3)}\varepsilon_{xz}^{(3)} + {}^{\mathrm{H}}\sigma_{zz}^{(3)}\varepsilon_{zz}^{(3)} + {}^{\mathrm{H}}\sigma_{yy}^{(3)}\underbrace{\varepsilon_{yy}^{(3)}}_{0} + 2\underbrace{{}^{\mathrm{H}}\sigma_{xy}^{(3)}}_{0}\underbrace{\varepsilon_{xy}^{(3)}}_{0} + 2\underbrace{{}^{\mathrm{H}}\sigma_{yz}^{(3)}}_{0}\underbrace{\varepsilon_{yz}^{(3)}}_{0}\right)\mathrm{d}V, \end{aligned} \tag{64b}$$

where V_1, V_2, V_3 are volumes of the lower face sheet, core and upper face sheet. The underbraced terms in the above expression are equal to zero due to the condition of plane strain. If Eqs. (57) and (61) are used, we can write the expression for the strain energy in the form

$$\begin{aligned} U &= \frac{1}{2}b\int_0^L\int_{z_1}^{z_2}\{\varepsilon^{(1)}\}^{\mathrm{T}}\{{}^{\mathrm{H}}\sigma^{(1)}\}\,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}b\int_0^L\int_{z_2}^{z_3}\{\varepsilon^{(2)}\}^{\mathrm{T}}\{{}^{\mathrm{H}}\sigma^{(2)}\}\,\mathrm{d}z\,\mathrm{d}x \\ &\quad + \frac{1}{2}b\int_0^L\int_{z_3}^{z_4}\{\varepsilon^{(3)}\}^{\mathrm{T}}\{{}^{\mathrm{H}}\sigma^{(3)}\}\,\mathrm{d}z\,\mathrm{d}x \\ &= \frac{1}{2}b\int_0^L\int_{z_1}^{z_2}\{\varepsilon^{(1)}\}^{\mathrm{T}}[C^{(1)}]\{\varepsilon^{(1)}\}\,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}b\int_0^L\int_{z_2}^{z_3}\{\varepsilon^{(2)}\}^{\mathrm{T}}[C^{(2)}]\{\varepsilon^{(2)}\}\,\mathrm{d}z\,\mathrm{d}x \\ &\quad + \frac{1}{2}b\int_0^L\int_{z_3}^{z_4}\{\varepsilon^{(3)}\}^{\mathrm{T}}[C^{(3)}]\{\varepsilon^{(3)}\}\,\mathrm{d}z\,\mathrm{d}x. \end{aligned} \tag{64c}$$

Eq. (56) is substituted into the last expression, yielding

$$\begin{aligned} U &= \frac{1}{2}b\int_0^L \underset{(1\times5)}{\{f^{(1)}(x)\}^{\mathrm{T}}}\left(\int_{z_1}^{z_2}\underset{(5\times3)}{[Z(z)]^{\mathrm{T}}}\underset{(3\times3)}{[C^{(1)}]}\underset{(3\times5)}{[Z(z)]}\,\mathrm{d}z\right)\underset{(5\times1)}{\{f^{(1)}(x)\}}\,\mathrm{d}x \\ &\quad + \frac{1}{2}b\int_0^L \underset{(1\times5)}{\{f^{(2)}(x)\}^{\mathrm{T}}}\left(\int_{z_2}^{z_3}\underset{(5\times3)}{[Z(z)]^{\mathrm{T}}}\underset{(3\times3)}{[C^{(2)}]}\underset{(3\times5)}{[Z(z)]}\,\mathrm{d}z\right)\underset{(5\times1)}{\{f^{(2)}(x)\}}\,\mathrm{d}x \\ &\quad + \frac{1}{2}b\int_0^L \underset{(1\times5)}{\{f^{(3)}(x)\}^{\mathrm{T}}}\left(\int_{z_3}^{z_4}\underset{(5\times3)}{[Z(z)]^{\mathrm{T}}}\underset{(3\times3)}{[C^{(3)}]}\underset{(3\times5)}{[Z(z)]}\,\mathrm{d}z\right)\underset{(5\times1)}{\{f^{(3)}(x)\}}\,\mathrm{d}x, \end{aligned} \tag{65}$$

or

$$\begin{aligned} U = \frac{1}{2}b\int_0^L \Bigg(& \underset{(1\times5)}{\{f^{(1)}(x)\}^{\mathrm{T}}}\underset{(5\times5)}{[D^{(1)}]}\underset{(5\times1)}{\{f^{(1)}(x)\}} + \underset{(1\times5)}{\{f^{(2)}(x)\}^{\mathrm{T}}}\underset{(5\times5)}{[D^{(2)}]}\underset{(5\times1)}{\{f^{(2)}(x)\}} \\ & + \underset{(1\times5)}{\{f^{(3)}(x)\}^{\mathrm{T}}}\underset{(5\times5)}{[D^{(3)}]}\underset{(5\times1)}{\{f^{(3)}(x)\}}\Bigg)\,\mathrm{d}x, \end{aligned} \tag{66}$$

where

$$\begin{aligned} [D^{(1)}] &= \int_{z_1}^{z_2}\underset{(5\times3)}{[Z(z)]^{\mathrm{T}}}\underset{(3\times3)}{[C^{(1)}]}\underset{(3\times5)}{[Z(z)]}\,\mathrm{d}z \\ &= \frac{E^{(1)}}{1+v}\begin{bmatrix} (1-v)\frac{z_1-z_2}{2v-1} & \frac{1}{2}(1-v)\frac{z_1^2-z_2^2}{2v-1} & \frac{1}{3}(1-v)\frac{z_1^3-z_2^3}{2v-1} & 0 & v\frac{z_1-z_2}{2v-1} \\ \frac{1}{2}(1-v)\frac{z_1^2-z_2^2}{2v-1} & \frac{1}{3}(1-v)\frac{z_1^3-z_2^3}{2v-1} & \frac{1}{4}(1-v)\frac{z_1^4-z_2^4}{2v-1} & 0 & \frac{1}{2}v\frac{z_1^2-z_2^2}{2v-1} \\ \frac{1}{3}(1-v)\frac{z_1^3-z_2^3}{2v-1} & \frac{1}{4}(1-v)\frac{z_1^4-z_2^4}{2v-1} & \frac{1}{5}(1-v)\frac{z_1^5-z_2^5}{2v-1} & 0 & \frac{1}{3}v\frac{z_1^3-z_2^3}{2v-1} \\ 0 & 0 & 0 & \frac{1}{2}(z_2-z_1) & 0 \\ v\frac{z_1-z_2}{2v-1} & \frac{1}{2}v\frac{z_1^2-z_2^2}{2v-1} & \frac{1}{3}v\frac{z_1^3-z_2^3}{2v-1} & 0 & (1-v)\frac{z_1-z_2}{2v-1} \end{bmatrix}, \end{aligned} \tag{67}$$

$$[D^{(2)}] = \int_{z_2}^{z_3} \underset{(5\times 3)}{[Z(z)]^{\mathrm{T}}} \underset{(3\times 3)}{[C^{(2)}]} \underset{(3\times 5)}{[Z(z)]} \,\mathrm{d}z$$

$$= \frac{E^{(2)}}{1+v} \begin{bmatrix} (1-v)\frac{z_2-z_3}{2v-1} & \frac{1}{2}(1-v)\frac{z_2^2-z_3^2}{2v-1} & \frac{1}{3}(1-v)\frac{z_2^3-z_3^3}{2v-1} & 0 & v\frac{z_2-z_3}{2v-1} \\ \frac{1}{2}(1-v)\frac{z_2^2-z_3^2}{2v-1} & \frac{1}{3}(1-v)\frac{z_2^3-z_3^3}{2v-1} & \frac{1}{4}(1-v)\frac{z_2^4-z_3^4}{2v-1} & 0 & \frac{1}{2}v\frac{z_2^2-z_3^2}{2v-1} \\ \frac{1}{3}(1-v)\frac{z_2^3-z_3^3}{2v-1} & \frac{1}{4}(1-v)\frac{z_2^4-z_3^4}{2v-1} & \frac{1}{5}(1-v)\frac{z_2^5-z_3^5}{2v-1} & 0 & \frac{1}{3}v\frac{z_2^3-z_3^3}{2v-1} \\ 0 & 0 & 0 & \frac{1}{2}(z_3-z_2) & 0 \\ v\frac{z_2-z_3}{2v-1} & \frac{1}{2}v\frac{z_2^2-z_3^2}{2v-1} & \frac{1}{3}v\frac{z_2^3-z_3^3}{2v-1} & 0 & (1-v)\frac{z_2-z_3}{2v-1} \end{bmatrix}, \tag{68}$$

$$[D^{(3)}] = \int_{z_3}^{z_4} \underset{(5\times 3)}{[Z(z)]^{\mathrm{T}}} \underset{(3\times 3)}{[C^{(3)}]} \underset{(3\times 5)}{[Z(z)]} \,\mathrm{d}z$$

$$= \frac{E^{(3)}}{1+v} \begin{bmatrix} (1-v)\frac{z_3-z_4}{2v-1} & \frac{1}{2}(1-v)\frac{z_3^2-z_4^2}{2v-1} & \frac{1}{3}(1-v)\frac{z_3^3-z_4^3}{2v-1} & 0 & v\frac{z_3-z_4}{2v-1} \\ \frac{1}{2}(1-v)\frac{z_3^2-z_4^2}{2v-1} & \frac{1}{3}(1-v)\frac{z_3^3-z_4^3}{2v-1} & \frac{1}{4}(1-v)\frac{z_3^4-z_4^4}{2v-1} & 0 & \frac{1}{2}v\frac{z_3^2-z_4^2}{2v-1} \\ \frac{1}{3}(1-v)\frac{z_3^3-z_4^3}{2v-1} & \frac{1}{4}(1-v)\frac{z_3^4-z_4^4}{2v-1} & \frac{1}{5}(1-v)\frac{z_3^5-z_4^5}{2v-1} & 0 & \frac{1}{3}v\frac{z_3^3-z_4^3}{2v-1} \\ 0 & 0 & 0 & \frac{1}{2}(z_4-z_3) & 0 \\ v\frac{z_3-z_4}{2v-1} & \frac{1}{2}v\frac{z_3^2-z_4^2}{2v-1} & \frac{1}{3}v\frac{z_3^3-z_4^3}{2v-1} & 0 & (1-v)\frac{z_3-z_4}{2v-1} \end{bmatrix}. \tag{69}$$

Expression (66) for the strain energy can be written in the form

$$U = \frac{1}{2}b\int_0^L \underset{(1\times 15)}{\begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}^{\mathrm{T}}} \underset{(15\times 1)}{\begin{Bmatrix} [D^{(1)}]\{f^{(1)}\} \\ [D^{(2)}]\{f^{(2)}\} \\ [D^{(3)}]\{f^{(3)}\} \end{Bmatrix}} \mathrm{d}x = \frac{1}{2}b\int_0^L \underset{(1\times 15)}{\begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}^{\mathrm{T}}} \underset{(15\times 15)}{\begin{bmatrix} \underset{(5\times 5)}{[D^{(1)}]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(2)}]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(3)}]} \end{bmatrix}} \underset{(15\times 1)}{\begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}} \mathrm{d}x,$$

or

$$U = \frac{1}{2}b\int_0^L \underset{(1\times 15)}{\{f\}^{\mathrm{T}}} \underset{(15\times 15)}{[D]} \underset{(15\times 1)}{\{f\}} \,\mathrm{d}x, \tag{70}$$

where

$$\underset{(1\times 15)}{\{f\}^{\mathrm{T}}} = \begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}, \tag{71}$$

$$\underset{(15\times 15)}{[D]} = \begin{bmatrix} \underset{(5\times 5)}{[D^{(1)}]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(2)}]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(3)}]} \end{bmatrix}. \tag{72}$$

8. Virtual work of external forces in terms of variations of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$

Virtual work of loads on the upper and lower surfaces, $q_{\rm u}$ and $q_{\rm l}$ correspondingly, is [1]

$$\delta' W = \int_0^L \left(q_{\rm l}\delta w|_{z=z_1} + q_{\rm u}\delta w|_{z=z_4} \right) {\rm d}x = \int_0^L \left(q_{\rm l}\delta w^{(1)}\big|_{z=z_1} + q_{\rm u}\delta w^{(3)}\big|_{z=z_4} \right) {\rm d}x. \tag{73}$$

According to Eqs. (35) and (36),

$$\delta w^{(1)}\big|_{z=z_1} = \delta w_0 + z_2\delta\varepsilon_{zz}^{(2)} + (z_1 - z_2)\delta\varepsilon_{zz}^{(1)}, \tag{74}$$

$$\delta w^{(3)}\big|_{z=z_4} = \delta w_0 + z_3\delta\varepsilon_{zz}^{(2)} + (z_4 - z_3)\delta\varepsilon_{zz}^{(3)}. \tag{75}$$

Eqs. (74) and (75) are substituted into Eq. (73) yielding

$$\delta' W = \int_0^L q_{\rm l}\left[\delta w_0 + z_2\delta\varepsilon_{zz}^{(2)} + (z_1 - z_2)\delta\varepsilon_{zz}^{(1)}\right] {\rm d}x + \int_0^L q_{\rm u}\left[\delta w_0 + z_3\delta\varepsilon_{zz}^{(2)} + (z_4 - z_3)\delta\varepsilon_{zz}^{(3)}\right] {\rm d}x. \tag{76}$$

9. Finite element formulation for static problem of cylindrical bending of the sandwich isotropic plate

The column-matrices $\{f^{(k)}\}$, defined by Eq. (59), can be written in the form

$$\underset{(5\times1)}{\{f^{(1)}\}} = \underset{(5\times8)}{[\partial_1]}\underset{(8\times1)}{\{F\}}, \tag{77}$$

$$\underset{(5\times1)}{\{f^{(2)}\}} = \underset{(5\times8)}{[\partial_2]}\underset{(8\times1)}{\{F\}}, \tag{78}$$

$$\underset{(5\times1)}{\{f^{(3)}\}} = \underset{(5\times8)}{[\partial_3]}\underset{(8\times1)}{\{F\}}, \tag{79}$$

or

$$\left\{ \begin{array}{c} \underset{(5\times1)}{\{f^{(1)}\}} \\ \underset{(5\times1)}{\{f^{(2)}\}} \\ \underset{(5\times1)}{\{f^{(3)}\}} \end{array} \right\} = \left[\begin{array}{c} \underset{(5\times8)}{[\partial_1]} \\ \underset{(5\times8)}{[\partial_2]} \\ \underset{(5\times8)}{[\partial_3]} \end{array} \right] \underset{(8\times1)}{\{F\}}, \tag{80}$$

[1] If at a coordinate x_0 there is a concentrated force P, applied, for example, to the upper surface of the plate, then Eq. (73) is still valid, because the distributed load, $q_{\rm u}$, due to the concentrated force can be formally written in the form $q_{\rm u} = P\Delta(x - x_0)$, where $\Delta(x - x_0)$ is the Dirac's delta-function. This function can be defined by the formula $\Delta(x - x_0) = 1/\pi \int_0^\infty \cos r(x - x_0)\,{\rm d}r$, and it has the following properties:

$$\Delta(x - x_0) = \begin{cases} \infty & \text{at } x = x_0, \\ 0 & \text{at } x \neq x_0, \end{cases}$$

$$\int_{X_1}^{X_2} f(x)\Delta(x - x_0)\,{\rm d}x = \begin{cases} f(x_0) & \text{if } X_1 < x < X_2, \\ \frac{1}{2}f(x_0) & \text{if } x_0 = X_1 \quad \text{or } x_0 = X_2, \\ 0 & \text{if } x < X_1 \quad \text{or } x > X_2. \end{cases}$$

where

$$\underset{(8\times 1)}{\{F\}} \equiv \begin{Bmatrix} u_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix} \tag{81}$$

is column-matrix of the unknown functions of the problem,

$$\underset{(5\times 8)}{[\partial_1]} \equiv \begin{bmatrix} \frac{d}{dx} & 0 & -2z_2\frac{d}{dx} & -\frac{1}{2}z_2^2\frac{d^2}{dx^2} & 2z_2\frac{d}{dx} & \frac{1}{2}z_2^2\frac{d^2}{dx^2} & 0 & 0 \\ 0 & -\frac{d^2}{dx^2} & 2\frac{d}{dx} & z_2\frac{d^2}{dx^2} & 0 & -z_2\frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{82}$$

$$\underset{(5\times 8)}{[\partial_2]} \equiv \begin{bmatrix} \frac{d}{dx} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d^2}{dx^2} & 0 & 0 & 2\frac{d}{dx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{83}$$

$$\underset{(5\times 8)}{[\partial_3]} \equiv \begin{bmatrix} \frac{d}{dx} & 0 & 0 & 0 & 2z_3\frac{d}{dx} & \frac{1}{2}z_3^2\frac{d^2}{dx^2} & -2z_3\frac{d}{dx} & -\frac{1}{2}z_3^2\frac{d^2}{dx^2} \\ 0 & -\frac{d^2}{dx^2} & 0 & 0 & 0 & -z_3\frac{d^2}{dx^2} & 2\frac{d}{dx} & z_3\frac{d^2}{dx^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{84}$$

Using notation of Eq. (71),

$$\underset{(15\times 1)}{\{f\}} \equiv \begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix},$$

and the notation

$$\underset{(15\times 8)}{[\partial]} \equiv \begin{bmatrix} \underset{(5\times 8)}{[\partial_1]} \\ \underset{(5\times 8)}{[\partial_2]} \\ \underset{(5\times 8)}{[\partial_3]} \end{bmatrix}, \tag{85}$$

we will write Eq. (80) in the form

$$\underset{(15\times 1)}{\{f\}} = \underset{(15\times 8)}{[\partial]}\underset{(8\times 1)}{\{F\}}. \tag{86}$$

Eq. (86) can be substituted into Eq. (70) for the strain energy resulting in

$$U = \frac{1}{2} b \int_0^L \left(\underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \right)^{\mathrm{T}} \underset{(15\times 15)}{[D]} \underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \,\mathrm{d}x. \tag{87}$$

The strain energy of a finite element is

$$\overline{U} = \frac{1}{2} b \int_{x_1}^{x_2} \left(\underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \right)^{\mathrm{T}} \underset{(15\times 15)}{[D]} \underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \,\mathrm{d}x = \frac{1}{2} b \int_0^l \left(\underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \right)^{\mathrm{T}} \underset{(15\times 15)}{[D]} \underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \,\mathrm{d}\overline{x}, \tag{88}$$

where x_1 and x_2 are coordinates of the end-points of a finite element in a global coordinate system; $\overline{x}$ is an x-coordinate in a local, element coordinate system, whose origin is at a left node of an element; $l = x_2 - x_1$ is a length of a finite element.

According to Eq. (76), the virtual work of external forces acting on a finite element of the plate is

$$\begin{aligned} \delta'\overline{W} &= \int_0^l \left[(q_1 + q_\mathrm{u})\delta w_0 + (z_1 - z_2) q_1 \delta\varepsilon_{zz}^{(1)} + (z_2 q_1 + z_3 q_\mathrm{u})\delta\varepsilon_{zz}^{(2)} + (z_4 - z_3) q_\mathrm{u} \delta\varepsilon_{zz}^{(3)} \right] \mathrm{d}\overline{x} \\ &= \int_0^l \begin{Bmatrix} \delta u_0 \\ \delta w_0 \\ \delta\varepsilon_{xz}^{(1)} \\ \delta\varepsilon_{zz}^{(1)} \\ \delta\varepsilon_{xz}^{(2)} \\ \delta\varepsilon_{zz}^{(2)} \\ \delta\varepsilon_{xz}^{(3)} \\ \delta\varepsilon_{zz}^{(3)} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ q_1 + q_\mathrm{u} \\ 0 \\ (z_1 - z_2) q_1 \\ 0 \\ (z_2 q_1 + z_3 q_\mathrm{u}) \\ 0 \\ (z_4 - z_3) q_\mathrm{u} \end{Bmatrix} \mathrm{d}\overline{x} = \int_0^l \left(\underset{(8\times 1)}{\delta\{F\}} \right)^{\mathrm{T}} \underset{(8\times 1)}{\{q\}} \,\mathrm{d}\overline{x}, \end{aligned} \tag{89}$$

where $\{F\}$ is defined by Eq. (81), and

$$\underset{(8\times 1)}{\{q\}} \equiv \lfloor 0 \quad (q_1 + q_\mathrm{u}) \quad 0 \quad (z_1 - z_2) q_1 \quad 0 \quad (z_2 q_1 + z_3 q_\mathrm{u}) \quad 0 \quad (z_4 - z_3) q_\mathrm{u} \rfloor^{\mathrm{T}}. \tag{90}$$

So, the principle of virtual work for a finite element, $\delta\overline{U} - \delta'\overline{W} = 0$, takes the form

$$\frac{1}{2} b \delta \int_0^l \underset{(1\times 15)}{\left(\underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \right)^{\mathrm{T}}} \underset{(15\times 15)}{[D]} \underset{(15\times 1)}{\underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}}} \,\mathrm{d}\overline{x} - \int_0^l \underset{(1\times 8)}{\left(\underset{(8\times 1)}{\delta\{F\}} \right)^{\mathrm{T}}} \underset{(8\times 1)}{\{q\}} \,\mathrm{d}\overline{x} = 0. \tag{91}$$

Now, for the finite element development, we need to represent the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ by interpolation polynomials. The maximum order of the derivatives of u_0 and of $\varepsilon_{xz}^{(k)}$ $(k = 1, 2, 3)$, entering into the virtual work Eq. (91) is 1. Therefore, the interpolation polynomials for u_0 and $\varepsilon_{xz}^{(k)}$ must be of at least first degree, and across boundaries between elements there must be continuity of u_0 (continuity of derivatives of u_0 and $\varepsilon_{xz}^{(k)}$ is not required). Therefore, we choose the first degree Lagrange polynomials to interpolate u_0 and $\varepsilon_{xz}^{(k)}$ $(k = 1, 2, 3)$ as functions of $\overline{x}$

$$u_0 = \lfloor M \rfloor \{\overline{u}\} = \lfloor M_1 \, M_2 \rfloor \{\overline{u}\}, \tag{92}$$

$$\varepsilon_{xz}^{(k)} = \lfloor M \rfloor \{\overline{\epsilon}^{(k)}\} = \lfloor M_1 \, M_2 \rfloor \{\overline{\epsilon}^{(k)}\}, \tag{93}$$

where

$$M_1 = 1 - \frac{\bar{x}}{l}, \qquad M_2 = \frac{\bar{x}}{l}, \tag{94}$$

$$\{\bar{u}\} = \begin{Bmatrix} u_0(0) \\ u_0(l) \end{Bmatrix}, \tag{95}$$

$$\{\bar{\epsilon}^{(k)}\} = \begin{Bmatrix} \varepsilon_{xz}^{(k)}(0) \\ \varepsilon_{xz}^{(k)}(l) \end{Bmatrix}. \tag{96}$$

The maximum order of the derivatives of w_0 and $\varepsilon_{zz}^{(k)}$ is 2. Therefore, interpolation polynomials for w_0 and $\varepsilon_{zz}^{(k)}$ must be of at least second degree and must have derivatives, continuous at the element boundaries up to the first order (i.e. w_0, $\mathrm{d}w_0/\mathrm{d}x$, $\varepsilon_{zz}^{(k)}$ and $\mathrm{d}\varepsilon_{zz}^{(k)}/\mathrm{d}x$ must be continuous). Therefore, we choose the Hermit polynomial of the third degree to interpolate w_0 and $\varepsilon_{zz}^{(k)}$

$$w_0 = \lfloor N \rfloor \{\bar{w}\} = \lfloor N_1 \; N_2 \; N_3 \; N_4 \rfloor \{\bar{w}\}, \tag{97}$$

$$\varepsilon_{zz}^{(k)} = \lfloor N \rfloor \{\bar{\varepsilon}^{(k)}\} = \lfloor N_1 \; N_2 \; N_3 \; N_4 \rfloor \{\bar{\varepsilon}^{(k)}\}, \tag{98}$$

where

$$N_1 = 1 - \frac{3\bar{x}^2}{l^2} + \frac{2\bar{x}^3}{l^3}, \quad N_2 = \bar{x} - \frac{2\bar{x}^2}{l} + \frac{\bar{x}^3}{l^2}, \quad N_3 = \frac{3\bar{x}^2}{l^2} - \frac{2\bar{x}^3}{l^3}, \quad N_4 = -\frac{\bar{x}^2}{l} + \frac{\bar{x}^3}{l^2}, \tag{99}$$

$$\{\bar{w}\} = \begin{Bmatrix} w_0(0) \\ \frac{\mathrm{d}w_0}{\mathrm{d}x}(0) \\ w_0(l) \\ \frac{\mathrm{d}w_0}{\mathrm{d}x}(l) \end{Bmatrix}, \tag{100}$$

$$\{\bar{\varepsilon}^{(k)}\} = \begin{Bmatrix} \varepsilon_{zz}^{(k)}(0) \\ \frac{\mathrm{d}\varepsilon_{zz}^{(k)}}{\mathrm{d}x}(0) \\ \varepsilon_{zz}^{(k)}(l) \\ \frac{\mathrm{d}\varepsilon_{zz}^{(k)}}{\mathrm{d}x}(l) \end{Bmatrix}. \tag{101}$$

The column-matrix $\{F\}$ of the unknown functions of the problem, defined by Eq. (81), now can be written in the form

$$\{F\} \equiv \begin{Bmatrix} u_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \lfloor M \rfloor \{\bar{u}\} \\ \lfloor N \rfloor \{\bar{w}\} \\ \lfloor M \rfloor \{\bar{\epsilon}^{(1)}\} \\ \lfloor N \rfloor \{\bar{\varepsilon}^{(1)}\} \\ \lfloor M \rfloor \{\bar{\epsilon}^{(2)}\} \\ \lfloor N \rfloor \{\bar{\varepsilon}^{(2)}\} \\ \lfloor M \rfloor \{\bar{\epsilon}^{(3)}\} \\ \lfloor N \rfloor \{\bar{\varepsilon}^{(3)}\} \end{Bmatrix} = \begin{bmatrix} \lfloor M \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor N \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor M \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor N \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor M \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor N \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor M \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor N \rfloor \end{bmatrix} \begin{Bmatrix} \{\bar{u}\} \\ \{\bar{w}\} \\ \{\bar{\epsilon}^{(1)}\} \\ \{\bar{\varepsilon}^{(1)}\} \\ \{\bar{\epsilon}^{(2)}\} \\ \{\bar{\varepsilon}^{(2)}\} \\ \{\bar{\epsilon}^{(3)}\} \\ \{\bar{\varepsilon}^{(3)}\} \end{Bmatrix}, \tag{102}$$

or

$$\underset{(8\times1)}{\{F\}} = \underset{(8\times24)}{[Q]} \; \underset{(24\times1)}{\{d\}}, \tag{103}$$

where

$$\underset{(8\times24)}{[Q]} \equiv \begin{bmatrix} \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} \end{bmatrix} \tag{104}$$

is a matrix of shape functions and

$$\underset{(24\times1)}{\{d\}} \equiv \begin{Bmatrix} \underset{(2\times1)}{\{\overline{u}\}} \\ \underset{(4\times1)}{\{\overline{w}\}} \\ \underset{(2\times1)}{\{\overline{\epsilon}^{(1)}\}} \\ \underset{(4\times1)}{\{\overline{\varepsilon}^{(1)}\}} \\ \underset{(2\times1)}{\{\overline{\epsilon}^{(2)}\}} \\ \underset{(4\times1)}{\{\overline{\varepsilon}^{(2)}\}} \\ \underset{(2\times1)}{\{\overline{\epsilon}^{(3)}\}} \\ \underset{(4\times1)}{\{\overline{\varepsilon}^{(3)}\}} \end{Bmatrix}, \tag{105a}$$

is a vector of nodal degrees of freedom of an element, where $\underset{(2\times1)}{\{\overline{u}\}}$ is defined by Eq. (95), $\underset{(4\times1)}{\{\overline{w}\}}$ is defined by Eq. (100), vectors $\underset{(2\times1)}{\{\overline{\epsilon}^{(1)}\}}$, $\underset{(2\times1)}{\{\overline{\epsilon}^{(2)}\}}$, $\underset{(2\times1)}{\{\overline{\epsilon}^{(3)}\}}$ are defined by formulas (96), and vectors $\underset{(4\times1)}{\{\overline{\varepsilon}^{(1)}\}}$, $\underset{(4\times1)}{\{\overline{\varepsilon}^{(2)}\}}$, $\underset{(4\times1)}{\{\overline{\varepsilon}^{(3)}\}}$ are defined by Eq. (101). Therefore, the vector of nodal degrees of freedom of an element is

$$\underset{(24\times1)}{\{d\}} \equiv \lfloor d_1 \quad d_2 \quad \ldots \quad d_{24} \rfloor^{\mathrm{T}}, \tag{105b}$$

where

$$
\begin{aligned}
&d_1 = u_0(0), \quad d_2 = u_0(l), \quad d_3 = w_0(0), \quad d_4 = w_0^{'}(0), \quad d_5 = w_0(l), \quad d_6 = w_0^{'}(l),\\
&d_7 = \varepsilon_{xz}^{(1)}(0), \quad d_8 = \varepsilon_{xz}^{(1)}(l), \quad d_9 = \varepsilon_{zz}^{(1)}(0), \quad d_{10} = \frac{\mathrm{d}\varepsilon_{zz}^{(1)}}{\mathrm{d}x}(0), \quad d_{11} = \varepsilon_{zz}^{(1)}(l),\\
&d_{12} = \frac{\mathrm{d}\varepsilon_{zz}^{(1)}}{\mathrm{d}x}(l), \quad d_{13} = \varepsilon_{xz}^{(2)}(0), \quad d_{14} = \varepsilon_{xz}^{(2)}(l), \quad d_{15} = \varepsilon_{zz}^{(2)}(0), \quad d_{16} = \frac{\mathrm{d}\varepsilon_{zz}^{(2)}}{\mathrm{d}x}(0),\\
&d_{17} = \varepsilon_{zz}^{(2)}(l), \quad d_{18} = \frac{\mathrm{d}\varepsilon_{zz}^{(2)}}{\mathrm{d}x}(l), \quad d_{19} = \varepsilon_{xz}^{(3)}(0), \quad d_{20} = \varepsilon_{xz}^{(3)}(l), \quad d_{21} = \varepsilon_{zz}^{(3)}(0),\\
&d_{22} = \frac{\mathrm{d}\varepsilon_{zz}^{(3)}}{\mathrm{d}x}(0), \quad d_{23} = \varepsilon_{zz}^{(3)}(l), \quad d_{24} = \frac{\mathrm{d}\varepsilon_{zz}^{(3)}}{\mathrm{d}x}(l) \quad \text{(see Fig. 2).}
\end{aligned} \tag{106}
$$

These are the *nodal degrees of freedom of a finite element*.

Let us write Eq. (88) for the strain energy of a finite element in terms of the nodal degrees of freedom

$$
\begin{aligned}
\overline{U} &= \frac{1}{2} b \int_0^l \Bigg(\underset{(15\times8)}{[\partial]} \underset{(8\times1)}{\{F(\bar{x})\}} \Bigg)^{\mathrm{T}} \underset{(15\times15)}{[D]} \underset{(15\times8)}{[\partial]} \underset{(8\times1)}{\{F(\bar{x})\}} \,\mathrm{d}\bar{x}\\
&= \frac{1}{2} b \int_0^l \Bigg(\underset{(15\times8)}{[\partial]} \underset{(8\times24)}{[Q(\bar{x})]} \underset{(24\times1)}{\{d\}} \Bigg)^{\mathrm{T}} \underset{(15\times15)}{[D]} \underset{(15\times8)}{[\partial]} \underset{(8\times24)}{[Q(\bar{x})]} \underset{(24\times1)}{\{d\}} \,\mathrm{d}\bar{x}\\
&= \frac{1}{2} \underset{(1\times24)}{\{d\}^{\mathrm{T}}} \left(b \int_0^l \underset{(24\times15)}{\Bigg(\underset{(15\times8)}{[\partial]} \underset{(8\times24)}{[Q(\bar{x})]} \Bigg)^{T}} \underset{(15\times15)}{[D]} \underset{(15\times8)}{[\partial]} \underset{(8\times24)}{[Q(\bar{x})]} \mathrm{d}\bar{x} \right) \underset{(24\times1)}{\{d\}},
\end{aligned}
$$

or

$$
\overline{U} = \frac{1}{2} \underset{(1\times24)}{\{d\}^{\mathrm{T}}} \underset{(24\times24)}{\left[\tilde{k}\right]} \underset{(24\times1)}{\{d\}}, \tag{107}
$$

where

$$
\underset{(24\times24)}{\left[\tilde{k}\right]} = b \int_0^l \underset{(24\times15)}{\Bigg(\underset{(15\times8)}{[\partial]} \underset{(8\times24)}{[Q(\bar{x})]} \Bigg)^{\mathrm{T}}} \underset{(15\times15)}{[D]} \underset{(15\times8)}{[\partial]} \underset{(8\times24)}{[Q(\bar{x})]} \,\mathrm{d}\bar{x}. \tag{108}
$$

Let us write Eq. (89) for the virtual work of external forces, acting on a finite element of the plate, in terms of variations of the nodal degrees of freedom

$$
\delta'\overline{W} = \int_0^l \Bigg(\delta \underset{(8\times1)}{\{F(x)\}} \Bigg)^{\mathrm{T}} \underset{(8\times1)}{\{q(\bar{x})\}} \,\mathrm{d}\bar{x} = \int_0^l \Bigg(\underset{(8\times24)}{[Q(\bar{x})]} \delta \underset{(24\times1)}{\{d\}} \Bigg)^{\mathrm{T}} \underset{(8\times1)}{\{q(x)\}} \,\mathrm{d}\bar{x} = \delta \underset{(1\times24)}{\{d\}^{\mathrm{T}}} \int_0^l \underset{(24\times8)}{[Q(\bar{x})]^{\mathrm{T}}} \underset{(8\times1)}{\{q(\bar{x})\}} \,\mathrm{d}\bar{x}
$$

or

$$
\delta'\overline{W} = \delta \underset{(1\times24)}{\{d\}^{\mathrm{T}}} \underset{(24\times1)}{\{\tilde{r}\}}, \tag{109}
$$

where

$$
\underset{(24\times1)}{\{\tilde{r}\}} = \int_0^l \underset{(24\times8)}{[Q(\bar{x})]^{\mathrm{T}}} \underset{(8\times1)}{\{q(\bar{x})\}} \,\mathrm{d}\bar{x}. \tag{110}
$$

The substitution of Eqs. (107) and (109) into the principle of virtual work for a finite element, $\delta\overline{U} - \delta'\overline{W} = 0$, yields:

$$0 = \delta\left(\frac{1}{2}\underset{(1\times 24)}{\{d\}^{\mathrm{T}}}\underset{(24\times 24)}{\left[\widetilde{k}\right]}\underset{(24\times 1)}{\{d\}}\right) - \delta\underset{(1\times 24)}{\{d\}^{\mathrm{T}}}\underset{(24\times 1)}{\{r\}} = \left(\delta\{d\}^{\mathrm{T}}\right)\left(\left[\widetilde{k}\right]\{d\} - \{r\}\right). \tag{111}$$

Therefore

$$\underset{(24\times 24)}{\left[\widetilde{k}\right]}\underset{(24\times 1)}{\{d\}} = \underset{(24\times 1)}{\{\widetilde{r}\}}. \tag{112}$$

This is the equilibrium equation for a finite element in terms of the nodal degrees of freedom. For convenience of representation of a load acting on a wide plate in cylindrical bending, let us divide the left-hand and the right-hand sides of Eq. (112) by b:

$$\frac{1}{b}\underset{(24\times 24)}{\left[\widetilde{k}\right]}\underset{(24\times 1)}{\{d\}} = \frac{1}{b}\underset{(24\times 1)}{\{\widetilde{r}\}},$$

or

$$\underset{(24\times 24)}{[k]}\underset{(24\times 1)}{\{d\}} = \underset{(24\times 1)}{\{r\}}, \tag{113}$$

where

$$\underset{(24\times 24)}{[k]} = \frac{1}{b}\underset{(24\times 24)}{\left[\widetilde{k}\right]} = \int_0^l \underset{(24\times 15)}{\left(\underset{(15\times 8)}{[\partial]}\underset{(8\times 24)}{[Q(\bar{x})]}\right)^{\mathrm{T}}}\underset{(15\times 15)}{[D]}\underset{(15\times 8)}{[\partial]}\underset{(8\times 24)}{[Q(\bar{x})]}\,\mathrm{d}\bar{x}, \tag{114}$$

$$\underset{(24\times 1)}{\{r\}} = \frac{1}{b}\underset{(24\times 1)}{\{\widetilde{r}\}} = \frac{1}{b}\int_0^l \underset{(24\times 8)}{[Q(\bar{x})]^{\mathrm{T}}}\underset{(8\times 1)}{\{q(\bar{x})\}}\,\mathrm{d}\bar{x}. \tag{115}$$

Matrices $\underset{(24\times 24)}{[k]}$ and $\underset{(24\times 1)}{\{r\}}$ are the *stiffness matrix and load vector* of a finite element. In Eqs. (114) and (115) matrix $[\partial]$ is defined by Eq. (85), matrix $[Q]$-by Eq. (104), matrix $[D]$-by Eq. (72), matrix $\{q\}$-by Eq. (90). The integrations, required in the calculation of the element stiffness matrix through Eq. (114), were performed in closed form using a program for symbolic computation. Some of these expressions for components of the element stiffness matrix are shown in Appendix A.

10. Second form of expressions for the transverse stresses in terms of u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$

After computing the unknown functions $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$ $(k = 1, 2, 3)$ as a result of solving the finite element equations, we can find displacements, strains and stresses in the plate as functions of x- and z-coordinates (there is no dependence on the y-coordinate because we consider cylindrical bending). The displacements can be computed by Eqs. (34)–(36) and (41)–(43), the in-plane strains $\varepsilon_{xx}^{(1)}$, $\varepsilon_{xx}^{(2)}$, $\varepsilon_{xx}^{(3)}$ – by Eqs. (44)–(46), the in-plane stresses $\sigma_{xx}^{(1)}$, $\sigma_{xx}^{(2)}$, $\sigma_{xx}^{(3)}$ – by Eqs. (60)–(64a). The first forms of expressions for the transverse stresses in terms of $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$ (Eqs. (60)–(64a)), i.e. expressions for the transverse stresses obtained from the constitutive relations, were used only for the purpose of expressing the strain energy in terms of the unknown functions, which was used for the finite element formulation. In order to compute the transverse stresses accurately, we will use the second form of expressions for the transverse stresses in terms of $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$ (denoted as $\sigma_{xz}^{(k)} \equiv \left(\sigma_{xz}^{(k)}\right)^{(\mathrm{II})}$, $\sigma_{zz}^{(k)} \equiv \left(\sigma_{zz}^{(k)}\right)^{(\mathrm{II})}$), obtained from the equilibrium Eqs. (1) and (2). As it was written previously, the second forms of the transverse stresses are more accurate than the first forms.

Let us find expressions for $\sigma_{xz,z}^{(k)}$ and $\sigma_{zz,z}^{(k)}$ by integration of equilibrium equations. (1) and (2). If integration of the first equilibrium equation for the lower face sheet of the sandwich plate ($k = 1$)

$$\sigma_{xx,x}^{(1)} + \sigma_{xz,z}^{(1)} = 0 \quad (\text{Eq. (1)})$$

is performed with respect to z in the direction from the lower surface of the plate to its upper surface, we receive

$$\sigma_{xz}^{(1)} = \underbrace{\sigma_{xz}^{(1)}\big|_{z=z_1}}_{0} - \int_{z_1}^{z} {}^{\mathrm{H}}\sigma_{xx,x}^{(1)}\,\mathrm{d}z \quad (z_1 \leqslant z \leqslant z_2), \tag{116}$$

where $\sigma_{xz}^{(1)}\big|_{z=z_1} = 0$ due to the first boundary condition (16). It is obtained from Eq. (116) that

$$\sigma_{xz}^{(1)}\big|_{z=z_2} = -\int_{z_1}^{z_2} {}^{\mathrm{H}}\sigma_{xx,x}^{(1)}\,\mathrm{d}z. \tag{117}$$

Integration of the first equilibrium equation for the core of the sandwich plate ($k = 2$),

$$\sigma_{xx,x}^{(2)} + \sigma_{xz,z}^{(2)} = 0 \quad (\text{Eq. (1)}),$$

from z_2 to z, where $z_2 \leqslant z \leqslant z_3$, yields

$$\sigma_{xz}^{(2)} = \sigma_{xz}^{(2)}\big|_{z=z_2} - \int_{z_2}^{z} {}^{\mathrm{H}}\sigma_{xx,x}^{(2)}\,\mathrm{d}z. \tag{118}$$

According to the conditions of continuity of the transverse stresses at the interfaces between the plies with different material properties (Eq. (18)) and according to Eq. (117), we have

$$\sigma_{xz}^{(2)}\big|_{z=z_2} = \sigma_{xz}^{(1)}\big|_{z=z_2} = -\int_{z_1}^{z_2} {}^{\mathrm{H}}\sigma_{xx,x}^{(1)}\,\mathrm{d}z. \tag{119}$$

If Eq. (119) is substituted into Eq. (118), one obtains

$$\sigma_{xz}^{(2)} = -\int_{z_1}^{z_2} {}^{\mathrm{H}}\sigma_{xx,x}^{(1)}\,\mathrm{d}z - \int_{z_2}^{z} {}^{\mathrm{H}}\sigma_{xx,x}^{(2)}\,\mathrm{d}z \quad (z_2 \leqslant z \leqslant z_3). \tag{120}$$

For the upper face sheet ($k = 3$), we receive analogously

$$\sigma_{xz}^{(3)} = -\int_{z_1}^{z_2} {}^{\mathrm{H}}\sigma_{xx,x}^{(1)}\,\mathrm{d}z - \int_{z_2}^{z_3} {}^{\mathrm{H}}\sigma_{xx,x}^{(2)}\,\mathrm{d}z - \int_{z_3}^{z} {}^{\mathrm{H}}\sigma_{xx,x}^{(3)}\,\mathrm{d}z \quad (z_3 \leqslant z \leqslant z_4). \tag{121}$$

Substitution of Eqs. (60)–(64a) for the in-plane stresses ${}^{\mathrm{H}}\sigma_{xx}^{(1)}$, ${}^{\mathrm{H}}\sigma_{xx}^{(2)}$, ${}^{\mathrm{H}}\sigma_{xx}^{(3)}$ in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{(zz)}^{(k)}$ into Eqs. (118), (121) and (122) yields the required second forms of expressions for the transverse stresses $\sigma_{xz}^{(k)} \equiv \left(\sigma_{xz}^{1}\right)^{(\mathrm{II})}$ in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$. These expressions are shown in Appendix B.

Integration of equilibrium equations (2)

$$\sigma_{xz,x}^{(k)} + \sigma_{zz,z}^{(k)} = 0 \quad (k = 1, 2, 3)$$

yields

$$\sigma_{zz}^{(1)} = \underbrace{\sigma_{zz}^{(1)}\big|_{z=z_1}}_{-\frac{q_1}{b}} - \int_{z_1}^{z} \sigma_{xz,x}^{(1)}\,\mathrm{d}z \quad (z_1 \leqslant z \leqslant z_2). \tag{122}$$

where $\sigma_{zz}^{(1)}\big|_{z=z_1} = -(q_1/b)$ due to a boundary condition, shown in Eq. (16),

$$\sigma_{zz}^{(2)} = -\frac{q_1}{b} - \int_{z_1}^{z_2} \sigma_{xz,x}^{(1)}\,\mathrm{d}z - \int_{z_2}^{z} \sigma_{xz,x}^{(2)}\,\mathrm{d}z \quad (z_2 \leqslant z \leqslant z_3). \tag{123}$$

$$\sigma_{zz}^{(3)} = -\frac{q_1}{b} - \int_{z_1}^{z_2} \sigma_{xz,x}^{(1)}\,\mathrm{d}z - \int_{z_2}^{z_3} \sigma_{xz,x}^{(2)}\,\mathrm{d}z - \int_{z_3}^{z} \sigma_{xz,x}^{(3)}\,\mathrm{d}z \quad (z_3 \leqslant z \leqslant z_4). \tag{124}$$

Substitution of expressions for $\sigma_{xz}^{(k)}$ in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ $(k = 1, 2, 3)$ into Eqs. (122)–(124) yields expressions for $\sigma_{zz}^{(k)} \equiv \left(\sigma_{zz}^{(k)}\right)^{(\mathrm{II})}$ in terms of the unknown functions. These expressions are shown in Appendix B.

Derivatives of the field variables u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ $(k = 1, 2, 3)$, that enter into expressions for the stresses $\sigma_{xx}^{(k)}$ (Eq. (60)), can be computed as derivatives of the interpolation polynomials for the field variables, that were used for the finite element formulation. The same method, applied to the computation of stresses $\sigma_{xz}^{(k)}$ from Eqs. (B.1)–(B.3) and stresses $\sigma_{zz}^{(k)}$ from Eqs. (B.4)–(B.6) can lead to wrong results. This is due to the fact that the equations of Appendix B for the stresses $\sigma_{xz}^{(k)}$ and $\sigma_{zz}^{(k)}$ contain derivatives of the field variables of the order higher than the degrees of the interpolation polynomials of these field variables in the finite element formulation, namely $\mathrm{d}^4w_0/\mathrm{d}x^4$, $\mathrm{d}^2u_0/\mathrm{d}x^2$, $\mathrm{d}^3u_0/\mathrm{d}x^3$, $\mathrm{d}^2\varepsilon_{xz}^{(k)}/\mathrm{d}x^2$, $\mathrm{d}^3\varepsilon_{xz}^{(k)}/\mathrm{d}x^3$ and $\mathrm{d}^4\varepsilon_{zz}^{(k)}/\mathrm{d}x^4$. Therefore, these derivatives, computed with the help of interpolation polynomials, used in the finite element formulation, vanish, that can be wrong for a particular problem. The use of the higher order interpolation polynomials in the finite element formulation can eliminate this difficulty, but this will lead to a significant increase in the number of degrees of freedom in the finite element model. Therefore, we computed these derivatives numerically, as $f((x_{i+1}) - f(x_i))/(x_{i+1} - x_i)$, using nodal values $f(x_i)$ of the field variables, obtained from the finite element solution. Let us consider, for example, computation of the derivative $\mathrm{d}^4w_0/\mathrm{d}x^4$, that enters into equations for $\left(\sigma_{zz}^{(k)}\right)^{(\mathrm{II})}$ (Appendix B, Eqs. (B.4), (B.5) and (B.6)). In the finite element formulation, the middle-surface transverse displacement, w_0, is approximated by the Hermite polynomial of the third degree. Therefore, the fourth derivative of this polynomial is equal to zero. On the other hand, in the example problem of a simply supported sandwich plate, uniformly loaded on the upper surface (Fig. 1b), that will be considered in Section 13, the values of function w_0 at the nodal points, computed by the finite element method when $L = 5$ m (with 10 elements) are

Node	1	2	3	4	5	6	7	8	9	10	11
w_0 (cm)	0.0000	−1.9730	−3.7328	−5.1105	−5.9854	−6.2851	−5.9854	−5.1105	−3.7328	−1.9730	0.0000

All finite elements in this computation had equal size. The values of the fourth derivative $\mathrm{d}^4w_0/\mathrm{d}x^4$, computed by applying a finite difference scheme to the column-matrix of w_0 (where x_i are the coordinates of the nodal points), are

Element	1	2	3	4	5	6	7
$\mathrm{d}^4w_0/\mathrm{d}x^4$	−0.7723	−0.7723	−0.7723	−0.7723	−0.7723	−0.7723	−0.7723

The numerical values, used in this example problem were: Young's modulus of the core $E^{(2)} = 1.0192 \times 10^8$ N/m^2, thickness of the core $t = 0.01$ m, Young's modulus of the face sheets $E^{(1)} = 1.9796 \times 10^{11}$ N/m^2, Poisson's ratio of the face sheets and the core $v = 0.3$, the total thickness of the plate $h = 0.02$ m, length of the plate $L = 5$ m, the externally applied normal pressure on the upper surface $q_u/b = -1 \times 10^5$ N/m^2. If we substitute these numerical values into the equation for d^4w_0/dx^4, obtained from the exact solution (Appendix C, Eq. (C.28)), we receive

$$\frac{d^4w_0}{dx^4} = -12\frac{q_u}{b}\frac{v^2-1}{E^{(1)}(h^3-t^3)+E^{(2)}t^3} = -0.7880\frac{1}{m^3}.$$

We see that the exact value of d^4w_0/dx^4 and the value computed numerically match well: the error is 2%. The drawback of this simplest method of numerical differentiation is that it does not allow one to compute the values of d^4w_0/dx^4 in the last three elements. This drawback can be overcome by using a more sophisticated method of numerical differentiation, for example by finding a least-squares polynomial approximation of the nodal values of the field variables and computing the derivatives of these polynomials. The numerically computed fourth-order derivative of a field variable, approximated within a finite element by a third-degree polynomial, is not equal to zero due to the fact that such a derivative represents a global variation of the field variable along the plate, not the local (within an element) variation, represented by the third-degree polynomial.

11. Satisfaction of boundary conditions on the upper surface of the plate

In the process of the deriving the second form of expressions for the transverse stresses in terms of the functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ (found in Appendix B), we used boundary conditions shown in Eq. (16) at the lower surface and conditions of continuity of the transverse stresses at the interfaces between the face sheets and the core (Eqs. (18) and (19)). Therefore, the second form of the transverse stresses satisfy these boundary and continuity conditions.

The boundary conditions on the upper surface, Eq. (17):

$$\sigma_{xz}^{(3)} = 0, \quad \sigma_{zz}^{(3)} = \frac{q_u}{b} \qquad \text{at } z = \frac{h}{2} = z_4,$$

will also be satisfied by the second forms of the transverse stresses. This can be shown as follows: if we substitute the second forms of expressions for $\sigma_{xz}^{(3)}$ and $\sigma_{zz}^{(3)}$ (Appendix B, Eqs. (B.3) and (B.6)) into the boundary conditions (17) on the upper surface, we obtain two differential equations for the functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$, that can also be obtained from the principle of minimum total potential energy with the usual variational procedures (these procedures are not shown here because of large size of the resulting equations). Therefore, the virtual work principle contains information that the second forms of the transverse stresses $\sigma_{xz}^{(k)}$ and $\sigma_{zz}^{(k)}$ satisfy the boundary conditions on the upper surface of the plate. Thus, the finite element formulation based on the virtual work principle guarantees that the second form of the transverse stress satisfies approximately the boundary conditions on the upper surface of the plate. If, upon refinement of the finite element mesh, the finite element solution converges to the exact solution, then the values of the transverse stresses on the upper surface converge to the values of externally applied loads.

12. Second form of expressions for the transverse strains in terms of the unknown functions

The first form of the transverse strains $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ $(k = 1, 2, 3)$ are the unknown functions of the problem, and they can be computed directly by the finite element method. The more accurate values of the transverse strains, the second form of the transverse strains, can be computed by substituting the second form of the transverse stresses, formulas (Appendix B) into the strain–stress relations, Eqs. (13) and (14)

$$\left(\varepsilon_{zz}^{(k)}\right)^{(\mathrm{II})} = \frac{1-v^2}{E^{(k)}}\left[\left(\sigma_{zz}^{(k)}\right)^{(\mathrm{II})} - \frac{v}{1-v}\,{}^{\mathrm{H}}\sigma_{xx}^{(k)}\right], \tag{125}$$

$$\left(\varepsilon_{xz}^{(k)}\right)^{(\mathrm{II})} = \frac{1+v}{E^{(k)}}\left(\sigma_{xz}^{(k)}\right)^{(\mathrm{II})} \quad (k=1,2,3) \tag{126}$$

The in-plane stresses ${}^{\mathrm{H}}\sigma_{xx}^{(k)}$, which enter into these formulas, are computed by formulas (64).

The second form of the transverse strains, unlike the first forms (the assumed transverse strains), vary in the thickness direction.

13. Comparison of results of the plate theory with exact elasticity solution for a simply supported isotropic sandwich plate in cylindrical bending under a uniform load on the upper surface

Let us consider the cylindrical bending of a symmetric sandwich plate with isotropic face sheets and the core. The upper surface of the plate is under a uniform load with intensity (force per unit length) q_{u}. Along the edges $x = 0, L$ the plate is simply supported (Fig. 1b). The Young's moduli of the face sheets are equal and will be denoted by E_1 and the Young's modulus of the core will be denoted by E_2. We will consider the Poisson ratio, v, to be the same for all layers.

A load vector of a finite element is defined by Eq. (115). Computations give the following result for the load vector:

$$r_1 = 0, \quad r_2 = 0, \quad r_3 = \frac{1}{2}l\frac{q_{\mathrm{u}}}{b}, \quad r_4 = \frac{1}{12}l^2\frac{q_{\mathrm{u}}}{b}, \quad r_5 = \frac{1}{2}l\frac{q_{\mathrm{u}}}{b}, \quad r_6 = -\frac{1}{12}l^2\frac{q_{\mathrm{u}}}{b}, \quad r_7 = 0, \quad r_8 = 0,$$

$$r_9 = 0, \quad r_{10} = 0, \quad r_{11} = 0, \quad r_{12} = 0, \quad r_{13} = 0, \quad r_{14} = 0, \quad r_{15} = \frac{1}{2}lz_3\frac{q_{\mathrm{u}}}{b}, \quad r_{16} = \frac{1}{2}l^2\frac{q_{\mathrm{u}}}{b}z_3,$$

$$r_{17} = \frac{1}{2}lz_3\frac{q_{\mathrm{u}}}{b}, \quad r_{18} = -\frac{1}{12}l^2\frac{q_{\mathrm{u}}}{b}z_3, \quad r_{19} = 0, \quad r_{20} = 0, \quad r_{21} = \frac{1}{2}l\frac{q_{\mathrm{u}}}{b}(z_4 - z_3),$$

$$r_{22} = \frac{1}{12}l^2\frac{q_{\mathrm{u}}}{b}(z_4 - z_3), \quad r_{23} = \frac{1}{2}l\frac{q_{\mathrm{u}}}{b}(z_4 - z_3), \quad r_{24} = -\frac{1}{12}l^2\frac{q_{\mathrm{u}}}{b}(z_4 - z_3).$$

As an example, let us consider a sandwich plate with steel face sheets and an isotropic core, made of foam (Fig. 1a). We assume the following properties of the face sheets and the core; core: Young's modulus, $E^{(2)} = 1.0192 \times 10^8$ N/m^2, $v = 0.3$, thickness, $t = 0.01$ m; face sheets: Young's modulus, $E^{(1)} = 1.9796 \times 10^{11}$ N/m^2, Poisson ratio, $v = 0.3$, thickness of each face sheet, $(h/2) - (t/2) = 0.005$ m.

The total thickness of the plate is $h = 0.02$ m. We will consider the lengths L of the plate, varying in the range from 0.3 to 3 m. In order to provide the condition of cylindrical bending, we assume that the width b of the plate is much higher than its length L: $b/L \to \infty$. The plate is under the load $q_{\mathrm{u}}/b = -1 \times 10^5$ N/m^2.

We will compare the stresses, obtained from the finite element solution, based on the plate theory, with the stresses from the exact elasticity solution, presented in Appendix C. In this comparison the transverse stresses σ_{xz} and σ_{zz} from the plate theory, is the second form of the transverse stresses, obtained by integration of equilibrium equations (1) and (2). As it was discussed in Section 10, the second form of the transverse stresses is more accurate than the first form (obtained from the constitutive equations), and their

Table 1
Stress σ_{xx} at $x = L/2$

L (m)	h/L	σ_{xx}, at $z = -h/2$ (N/m^2)		σ_{xx}, at $z = 0$ (N/m^2)		σ_{xx}, at $z = h/2$ (N/m^2)	
		Exact	Plate theory	Exact	Plate theory	Exact	Plate theory
0.3	0.07	0.1929×10^8	0.2001×10^8 (error 3.7%)	0	−6.857	-0.1929×10^8	-0.2001×10^8 (error 3.7%)
0.5	0.04	0.5357×10^8	0.5540×10^8 (error 3.8%)	0	−6.857	-0.5357×10^8	-0.5540×10^8 (error 3.8%)
0.8	0.025	1.3714×10^8	1.4262×10^8 (error 4%)	0	−6.857	-1.3714×10^8	-1.4262×10^8 (error 4%)
1	0.02	2.1428×10^8	2.2306×10^8 (error 4.1%)	0	−6.857	-2.1428×10^8	-2.2264×10^8 (error 3.9%)
3	0.007	19.284×10^8	20.094 (error 4.2%)	0	−6.857	-19.284×10^8	−20.056 (error 4%)

Table 2
Stress σ_{xz} at $x = L/50$

L (m)	h/L	$\sigma_{xz} \times 10^6$ N/m^2 (at $z = (z_1 + z_2)/2$)		$\sigma_{xz}\|_{z=z_2} \times 10^6$ N/m^2		$\sigma_{xz}\|_{z=z_3} \times 10^6$ N/m^2		$\sigma_{xz}\|_{z=z_4} \times 10^6$ N/m^2	
		Exact	Plate theory	Exact	Plate theory	Exact	Plate theory	Exact	Plate theory
0.3	0.07	−0.562	−0.551 (error 1.9%)	−0.964	−0.945 (error 2%)	−0.964	−0.945 (error 2%)	0	-1.9×10^{-4}
0.5	0.04	−0.937	−0.919 (error 1.9%)	−1.607	−1.575 (error 2%)	−1.607	−1.575 (error 2%)	0	-3.5×10^{-4}
0.8	0.025	−1.500	−1.470 (error 2%)	−2.571	−2.520 (error 2%)	−2.571	−2.520 (error 2%)	0	2.0×10^{-4}
1	0.02	−1.875	−1.837 (error 2%)	−3.214	−3.150 (error 2%)	−3.214	−3.150 (error 2%)	0	9.2×10^{-5}
3	0.007	−5.624	−5.512 (error 2%)	−9.642	−9.449 (error 2%)	−9.642	−9.449 (error 2%)	0	1.8×10^{-4}

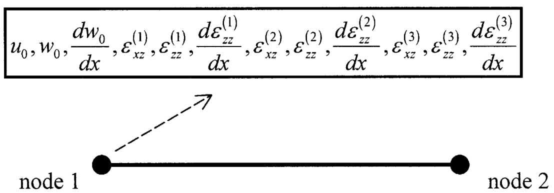

Fig. 2. Nodal variables, associated with one node of a finite element.

expressions in terms of the field variables u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ are given in Appendix B. The results of comparison for this example are presented in the Tables 1 and 2, and in the graphs in Figs. 2–5. Variation of stress σ_{xx} in the thickness direction is shown in Fig. 3. Variation of stress σ_{xz} in the thickness direction is shown in Fig. 4. The stress σ_{zz}, computed from the plate theory, like the one computed from the exact solution, does not depend on the length of the plate and does not vary along the length. Therefore, we show only the graph of the variation of this stress in the thickness direction, Fig. 4.

This comparison shows that the error in computation of the longitudinal stress σ_{xx} from the plate theory depends slightly on the aspect ratio (the ratio of thickness of the sandwich plate to its length): the lower the aspect ratio, the higher is error in computation of stress σ_{xx}. But this increase in the error is very small: as we see from the Table 1, the 10 times decrease of the thickness-to-length ratio of the plate leads to less than 1%

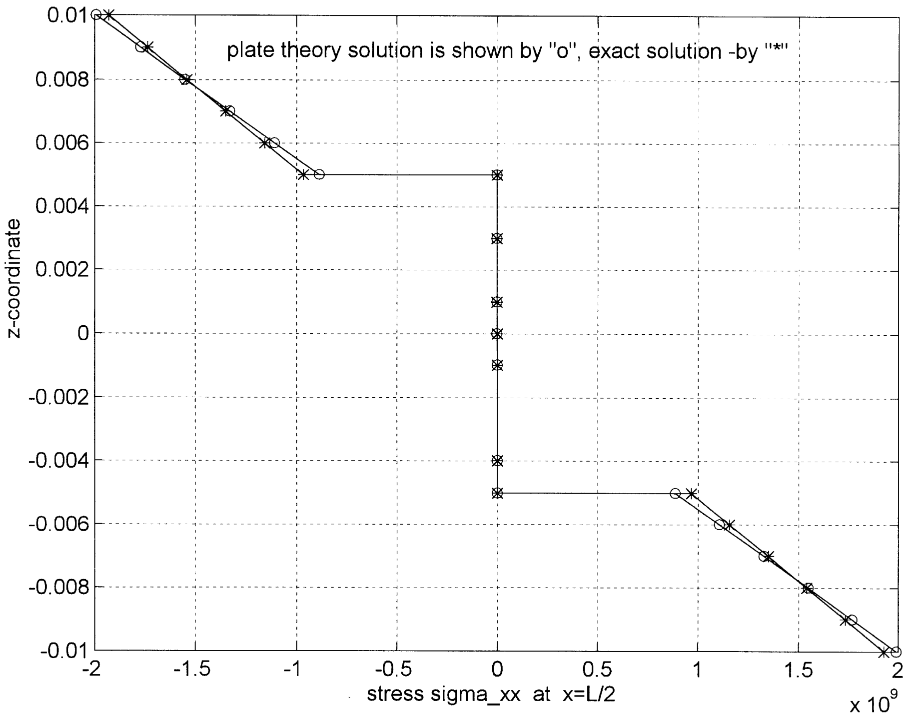

Fig. 3. Variation of stress, σ_{xx}, in the thickness direction, $L = 3$ m, $h = 0.02$ m.

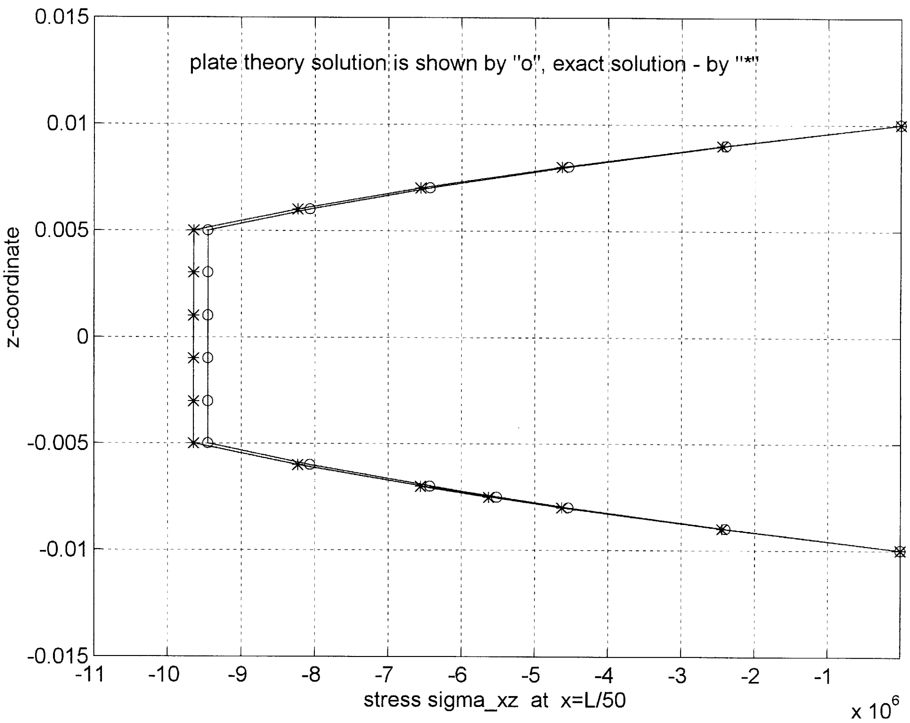

Fig. 4. Variation of stress σ_{xz} in the thickness direction, $L = 3$ m, $h = 0.02$ m.

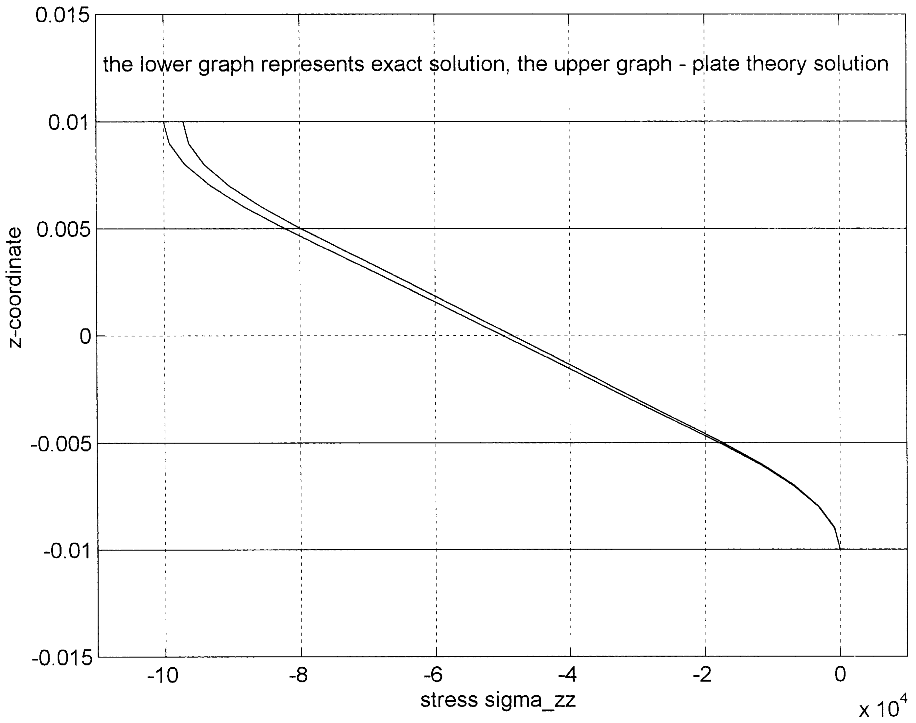

Fig. 5. Variation of stress σ_{zz} in the thickness direction, $L = 3$ m, $h = 0.02$ m.

increase of the error in computation of stress σ_{xx}. Therefore, we can make a conclusion that the shear locking phenomenon is not present in the developed finite element formulation.

The accuracy of results for the transverse stresses σ_{xz} and σ_{zz} depends very little on the aspect ratio. In the very wide range of aspect ratios, the error of computation of all stresses does not exceed 5%.

An idea discussed in Section 11, that the second form of the transverse stresses σ_{xz} and σ_{zz} (obtained by integration of equilibrium equations (1) and (2) and expressed in terms of the field variables by the formulas of Appendix B) satisfy approximately the boundary conditions on the upper surface is confirmed (in addition to satisfying exactly the boundary conditions on the lower surface and continuity conditions at the interfaces between the face sheets and the core). In this example, the discrepancy between the externally applied stress σ_{zz} on the upper surface and the computed second form of stress σ_{zz} on the upper surface is about 3% (Fig. 5). The externally applied stress σ_{xz} in this example is equal to zero, and the value of the second form of the stress σ_{xz} on the upper surface is a very small number as compared to the value of σ_{xz} at the interface between the upper face sheet and the core (Table 2 and Fig. 4). The second form of the transverse stresses satisfies the boundary conditions on the upper surface only approximately because it is computed by the formulas of Appendix B, in which we substituted the approximate values of field variables, computed by the finite element method.

In Fig. 6, the middle surface transverse displacements obtained from exact and finite element solutions are compared. The error is within the admissable limits, i.e. does not exceed 5%. The transverse stresses computed from the constitutive equations (the first form of the transverse stresses) do not satisfy the requirement of continuity across the boundaries between the face sheets and the core, and on the upper and

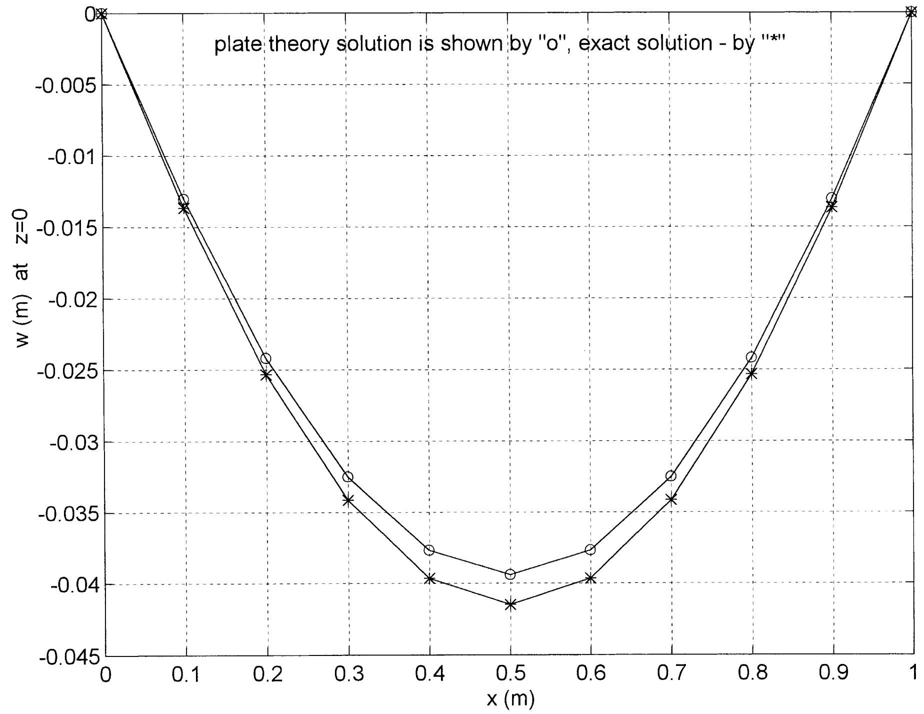

Fig. 6. Transverse displacement of the middle surface of the plate, $L = 1$ m.

lower surfaces they are not equal the externally applied loads. Therefore, the transverse stresses, computed from the constitutive equations, are highly inaccurate and can not be used, for example, in failure criteria.

14. Conclusions

The model of a sandwich plate, presented in this paper, is recommended for use in evaluating the stresses and deformations, if the sandwich plates have thick faces and transversely flexible cores, if they are loaded by concentrated or partially distributed forces, if they are placed on elastic foundation, and if the high accuracy of computation of the transverse stresses is required. A discrete-layer theory of a sandwich plate with the isotropic face sheets, in cylindrical bending, presented in this paper, based on assumed transverse strains and on subsequent improvement of the computed values of the transverse stresses and transverse strains with the help of equilibrium equations $\sigma_{ij,j} = 0$, produces sufficiently accurate results. The high accuracy of computation of the transverse stresses is achieved without resorting to the three-dimensional finite element analysis or the generalized layerwise laminated plate theory, based on assumed displacements, that require a larger number of degrees of freedom in the finite element models to achieve the similar accuracy. The shear locking phenomenon is not present in the developed finite element formulation. Therefore this approach for constructing a model of the sandwich plate deserves further study with regard to extension to two-dimensional models, models of sandwich plates with laminated composite face sheets, and also to dynamic and non-linear models.

Acknowledgements

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Appendix A. A few components of the stiffness matrix

Here we show only a few components of the stiffness matrix.

$$k_{11} = E^{(1)} \frac{z_1 - z_2 - v^{(1)} z_1 + v^{(1)} z_2}{l(1+v^{(1)})(2v^{(1)}-1)} - E^{(2)} \frac{-z_2 + z_3 + v^{(2)} z_2 - v^{(2)} z_3}{l(1+v^{(2)})(2v^{(2)}-1)} - E^{(3)} \frac{-z_3 + z_4 + v^{(3)} z_3 - v^{(3)} z_4}{l(1+v^{(3)})(2v^{(3)}-1)}, \tag{A.1}$$

$$k_{12} = -E^{(1)} \frac{z_1 - z_2 - v^{(1)} z_1 + v^{(1)} z_2}{l(1+v^{(1)})(2v^{(1)}-1)} + E^{(2)} \frac{-z_2 + z_3 + v^{(2)} z_2 - v^{(2)} z_3}{l(1+v^{(2)})(2v^{(2)}-1)} + E^{(3)} \frac{-z_3 + z_4 + v^{(3)} z_3 - v^{(3)} z_4}{l(1+v^{(3)})(2v^{(3)}-1)}, \tag{A.2}$$

$$k_{13} = 0, \tag{A.3}$$

$$k_{14} = -\frac{1}{2} E^{(1)} \frac{z_1^2 - z_2^2 - v^{(1)} z_1^2 + v^{(1)} z_2^2}{l(1+v^{(1)})(2v^{(1)}-1)} + \frac{1}{2} E^{(2)} \frac{-z_2^2 + z_3^2 + v^{(2)} z_2^2 - v^{(2)} z_3^2}{l(1+v^{(2)})(2v^{(2)}-1)} + \frac{1}{2} E^{(3)} \frac{-z_3^2 + z_4^2 + v^{(3)} z_3^2 - v^{(3)} z_4^2}{l(1+v^{(3)})(2v^{(3)}-1)}, \tag{A.4}$$

$$k_{15} = 0, \tag{A.5}$$

$$k_{16} = \frac{1}{2} E^{(1)} \frac{z_1^2 - z_2^2 - v^{(1)} z_1^2 + v^{(1)} z_2^2}{l(1+v^{(1)})(2v^{(1)}-1)} - \frac{1}{2} E^{(2)} \frac{-z_2^2 + z_3^2 + v^{(2)} z_2^2 - v^{(2)} z_3^2}{l(1+v^{(2)})(2v^{(2)}-1)} - \frac{1}{2} E^{(3)} \frac{-z_3^2 + z_4^2 + v^{(3)} z_3^2 - v^{(3)} z_4^2}{l(1+v^{(3)})(2v^{(3)}-1)}, \tag{A.6}$$

$$k_{17} = -E^{(1)} \frac{2 z_2 z_1 - z_2^2 - 2 z_2 v^{(1)} z_1 + v^{(1)} z_2^2 - z_1^2 + v^{(1)} z_1^2}{l(1+v^{(1)})(2v^{(1)}-1)}, \tag{A.7}$$

$$k_{18} = E^{(1)} \frac{2 z_2 z_1 - z_2^2 - 2 z_2 v^{(1)} z_1 + v^{(1)} z_2^2 - z_1^2 + v^{(1)} z_1^2}{l(1+v^{(1)})(2v^{(1)}-1)}. \tag{A.8}$$

Appendix B. Second form of the transverse stress components, evaluated from Eqs. (117), (120) and (121) in terms of the unknown functions ε_{xz}, ε_{zz}, u_0, w_0

$$\left(\sigma_{xz}^{(1)}\right)^{(\mathrm{II})} \equiv \sigma_{xz}^{(1)} = \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[u_{0,xx} + 2z_2\left(\varepsilon_{xz,xx}^{(2)} - \varepsilon_{xz,xx}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,xxx}^{(2)} - \varepsilon_{zz,xxx}^{(1)}\right)\right](z_1 - z) + \frac{E^{(1)}}{1+\nu^{(1)}} \times \frac{\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon_{zz,x}^{(1)}(z_1 - z) + \frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\varepsilon_{xz,xx}^{(1)} - w_{0,xxx} + z_2\left(\varepsilon_{zz,xxx}^{(1)} - \varepsilon_{zz,xxx}^{(2)}\right)\right] \times \left(z_1^2 - z^2\right) - \frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon_{zz,xxx}^{(1)}\left(z_1^3 - z^3\right), \tag{B.1}$$

$$\left(\sigma_{xz}^{(2)}\right)^{(\mathrm{II})} \equiv \sigma_{xz}^{(2)} = \frac{E^{(1)}}{1+\nu}\frac{1-\nu}{1-2\nu}\left[u_{0,xx} + 2z_2\left(\varepsilon_{xz,xx}^{(2)} - \varepsilon_{xz,xx}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,xxx}^{(2)} - \varepsilon_{zz,xxx}^{(1)}\right)\right](z_1 - z_2) + \frac{E^{(1)}}{1+\nu} \times \frac{\nu}{1-2\nu}\varepsilon_{zz,x}^{(1)}(z_1 - z_2) + \frac{1}{2}\frac{E^{(1)}}{1+\nu}\frac{1-\nu}{1-2\nu}\left[2\varepsilon_{xz,xx}^{(1)} - w_{0,xxx} + z_2\left(\varepsilon_{zz,xxx}^{(1)} - \varepsilon_{zz,xxx}^{(2)}\right)\right]\left(z_1^2 - z_2^2\right) - \frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon_{zz,xxx}^{(1)}\left(z_1^3 - z_2^3\right) + \frac{E^{(2)}}{(1-2\nu^{(2)})(1+\nu^{(2)})}\left[\left(1-\nu^{(2)}\right)u_{0,xx} + \nu^{(2)}\varepsilon_{zz,x}^{(2)}\right] \times (z_2 - z) + \frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\varepsilon_{xz,xx}^{(2)} - w_{0,xxx}\right)\left(z_2^2 - z^2\right) - \frac{1}{3}\varepsilon_{zz,xxx}^{(2)}\left(z_2^3 - z^3\right)\right], \tag{B.2}$$

$$\left(\sigma_{xz}^{(3)}\right)^{(\mathrm{II})} \equiv \sigma_{xz}^{(3)} = \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[u_{0,xx} + 2z_2\left(\varepsilon_{xz,xx}^{(2)} - \varepsilon_{xz,xx}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,xxx}^{(2)} - \varepsilon_{zz,xxx}^{(1)}\right)\right](z_1 - z_2) + \frac{E^{(1)}}{1+\nu^{(1)}} \times \frac{\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon_{zz,x}^{(1)}(z_1 - z_2) + \frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\varepsilon_{xz,xx}^{(1)} - w_{0,xxx} + z_2\left(\varepsilon_{zz,xxx}^{(1)} - \varepsilon_{zz,xxx}^{(2)}\right)\right] \times \left(z_1^2 - z_2^2\right) - \frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon_{zz,xxx}^{(1)}\left(z_1^3 - z_2^3\right) + \frac{E^{(2)}}{(1-2\nu^{(2)})(1+\nu^{(2)})} \times \left[\left(1-\nu^{(2)}\right)u_{0,xx} + \nu^{(2)}\varepsilon_{zz,x}^{(2)}\right](z_2 - z_3) + \frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\varepsilon_{xz,xx}^{(2)} - w_{0,xxx}\right)\left(z_2^2 - z_3^2\right) - \frac{1}{3}\varepsilon_{zz,xxx}^{(2)}\left(z_2^3 - z_3^3\right)\right] + \frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[u_{0,xx} + 2z_3\left(\varepsilon_{xz,xx}^{(2)} - \varepsilon_{xz,xx}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,xxx}^{(2)} - \varepsilon_{zz,xxx}^{(3)}\right)\right] \times (z_3 - z) + \frac{E^{(3)}}{1+\nu^{(3)}}\frac{\nu^{(3)}}{1-2\nu^{(3)}}\varepsilon_{zz,x}^{(3)}(z_3 - z) + \frac{1}{2}\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[2\varepsilon_{xz,xx}^{(3)} - w_{0,xxx} + z_3\left(\varepsilon_{zz,xxx}^{(3)} - \varepsilon_{zz,xxx}^{(2)}\right)\right]\left(z_3^2 - z^2\right) - \frac{1}{6}\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\varepsilon_{zz,xxx}^{(3)}\left(z_3^3 - z^3\right), \tag{B.3}$$

$$\left(\sigma_{zz}^{(1)}\right)^{(\mathrm{II})} \equiv \sigma_{zz}^{(1)}$$

$$= -\frac{q_1}{b} + \frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}(z-z_1)^2\left[\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3} + 2z_2\left(\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3}\right)\right.$$

$$\left. + \frac{1}{4}z_2^2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\right)\right] + \frac{1}{2}\frac{E^{(1)}}{1+v^{(1)}}\frac{v^{(1)}}{1-2v^{(1)}}(z-z_1)^2\frac{\mathrm{d}^2\varepsilon_{zz}^{(1)}}{\mathrm{d}x^2} + \frac{1}{6}\frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}$$

$$\times (z+2z_1)(z-z_1)^2\left[2\frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4} + z_2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\right)\right] - \frac{1}{24}\frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}$$

$$\times \left(z^2+2z_1z+3z_1^2\right)(z-z_1)^2\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}, \tag{B.4}$$

$$\left(\sigma_{zz}^{(2)}\right)^{(\mathrm{II})} \equiv \sigma_{zz}^{(2)}$$

$$= -\frac{q_1}{b} + \frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}(z_2-z_1)^2\left[\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3} + 2z_2\left(\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3}\right) + \frac{1}{4}z_2^2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\right)\right]$$

$$+ \frac{1}{2}\frac{E^{(1)}}{1+v^{(1)}}\frac{v^{(1)}}{1-2v^{(1)}}(z_2-z_1)^2\frac{\mathrm{d}^2\varepsilon_{zz}^{(1)}}{\mathrm{d}x^2} + \frac{1}{6}\frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}(z_2+2z_1)(z_2-z_1)^2$$

$$\times \left[2\frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4} + z_2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\right)\right] - \frac{1}{24}\frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}\left(z_2^2+2z_1z_2+3z_1^2\right)$$

$$\times (z_2-z_1)^2\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} + \frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}\left[\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3} + 2z_2\left(\varepsilon_{xz,xx}^{(2)} - \varepsilon_{xz,xx}^{(1)}\right) + \frac{1}{2}z_2^2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\right)\right]$$

$$\times (z_1-z_2)(z_2-z) + \frac{E^{(1)}}{1+v^{(1)}}\frac{v^{(1)}}{1-2v^{(1)}}\frac{\mathrm{d}^2\varepsilon_{zz}^{(1)}}{\mathrm{d}x^2}(z_1-z_2)(z_2-z) + \frac{1}{2}\frac{E^{(1)}}{1+v^{(1)}}\frac{1-v^{(1)}}{1-2v^{(1)}}$$

$$\times \left[2\frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4} + z_2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\right)\right]\left(z_1^2-z_2^2\right)(z_2-z) - \frac{1}{6}\frac{E^{(1)}}{1+v^{(1)}}$$

$$\times \frac{1-v^{(1)}}{1-2v^{(1)}}\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\left(z_1^3-z_2^3\right)(z_2-z) + \frac{E^{(2)}}{(1-2v^{(2)})(1+v^{(2)})}\left[\left(1-v^{(1)}\right)\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3}\right.$$

$$\left. + v^{(1)}\frac{\mathrm{d}^2\varepsilon_{zz}^{(2)}}{\mathrm{d}x^2}\right]\frac{1}{2}(z_2-z)^2 + \frac{1}{2}\frac{E^{(2)}}{1+v^{(2)}}\frac{1-v^{(2)}}{1-2v^{(2)}}\left[\left(2\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4}\right)\frac{1}{3}(2z_2+z)(z_2-z)^2\right.$$

$$\left. - \frac{1}{3}\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\frac{1}{4}\left(3z_2^2+2z_2z+z^2\right)(z_2-z)^2\right], \tag{B.5}$$

$$
\begin{aligned}
\left(\sigma_{zz}^{(3)}\right)^{(\mathrm{II})} &\equiv \sigma_{zz}^{(3)} \\
&= -\frac{q_1}{b} + \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}(z_2-z_1)^2\left[\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3} + 2z_2\left(\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3}\right) + \frac{1}{4}z_2^2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\right)\right] \\
&\quad + \frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}(z_2-z_1)^2\frac{\mathrm{d}^2\varepsilon_{zz}^{(1)}}{\mathrm{d}x^2} + \frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}(z_2+2z_1)(z_2-z_1)^2 \\
&\quad \times \left[2\frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4} + z_2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\right)\right] - \frac{1}{24}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z_2^2+2z_1z_2+3z_1^2\right) \\
&\quad \times (z_2-z_1)^2\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} + \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3} + 2z_2\left(\varepsilon_{xz,xx}^{(2)} - \varepsilon_{xz,xx}^{(1)}\right) + \frac{1}{2}z_2^2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\right)\right] \\
&\quad \times (z_1-z_2)(z_2-z_3) + \frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\frac{\mathrm{d}^2\varepsilon_{zz}^{(1)}}{\mathrm{d}x^2}(z_1-z_2)(z_2-z_3) + \frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}} \\
&\quad \times \frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4} + z_2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\right)\right]\left(z_1^2-z_2^2\right)(z_2-z_3) - \frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}} \\
&\quad \times \frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\left(z_1^3-z_2^3\right)(z_2-z_3) + \frac{E^{(2)}}{(1-2\nu^{(2)})(1+\nu^{(2)})}\left[\left(1-\nu^{(2)}\right)\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3}\right. \\
&\quad \left. + \nu^{(2)}\frac{\mathrm{d}^2\varepsilon_{zz}^{(2)}}{\mathrm{d}x^2}\right]\frac{1}{2}(z_2-z_3)^2 + \frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4}\right)\frac{1}{3}(2z_2+z_3)(z_2-z_3)^2\right. \\
&\quad \left. - \frac{1}{3}\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\frac{1}{4}\left(3z_2^2+2z_2z_3+z_3^2\right)(z_2-z_3)^2\right] + \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3}\right. \\
&\quad \left. + 2z_2\left(\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3}\right) + \frac{1}{2}z_2^2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\right)\right](z_1-z_2)(z_3-z) + \frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}} \\
&\quad \times \frac{\mathrm{d}^2\varepsilon_{zz}^{(1)}}{\mathrm{d}x^2}(z_1-z_2)(z_3-z) + \frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\frac{\mathrm{d}^3\varepsilon_{xz}^{(1)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4} + z_2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\right)\right] \\
&\quad \times \left(z_1^2-z_2^2\right)(z_3-z) - \frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\frac{\mathrm{d}^4\varepsilon_{zz}^{(1)}}{\mathrm{d}x^4}\left(z_1^3-z_2^3\right)(z_3-z) \\
&\quad + \frac{E^{(2)}}{(1-2\nu^{(2)})(1+\nu^{(2)})}\left[\left(1-\nu^{(2)}\right)\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3} + \nu\frac{\mathrm{d}^2\varepsilon_{zz}^{(2)}}{\mathrm{d}x^2}\right](z_2-z_3)(z_3-z) + \frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}} \\
&\quad \times \frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4}\right)\left(z_2^2-z_3^2\right) - \frac{1}{3}\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\left(z_2^3-z_3^3\right)\right](z_3-z) + \frac{E^{(3)}}{1+\nu^{(3)}} \\
&\quad \times \frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[\frac{\mathrm{d}^3u_0}{\mathrm{d}x^3} + 2z_3\left(\frac{\mathrm{d}^3\varepsilon_{xz}^{(2)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^3\varepsilon_{xz}^{(3)}}{\mathrm{d}x^3}\right) + \frac{1}{2}z_3^2\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(3)}}{\mathrm{d}x^4}\right)\right]\frac{1}{2}(z_3-z)^2 \\
&\quad + \frac{E^{(3)}}{1+\nu^{(3)}}\frac{\nu^{(3)}}{1-2\nu^{(3)}}\frac{\mathrm{d}^2\varepsilon_{zz}^{(3)}}{\mathrm{d}x^2}\frac{1}{2}(z_3-z)^2 + \frac{1}{2}\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[2\frac{\mathrm{d}^3\varepsilon_{xz}^{(3)}}{\mathrm{d}x^3} - \frac{\mathrm{d}^4w_0}{\mathrm{d}x^4}\right. \\
&\quad \left. + z_3\left(\frac{\mathrm{d}^4\varepsilon_{zz}^{(3)}}{\mathrm{d}x^4} - \frac{\mathrm{d}^4\varepsilon_{zz}^{(2)}}{\mathrm{d}x^4}\right)\right]\frac{1}{3}(2z_3+z)(z_3-z)^2 - \frac{1}{6}\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\frac{\mathrm{d}^4\varepsilon_{zz}^{(3)}}{\mathrm{d}x^4} \\
&\quad \times \frac{1}{4}\left(3z_3^2+2z_3z+z^2\right)(z_3-z)^2. \qquad \text{(B.6)}
\end{aligned}
$$

Appendix C. Exact elasticity solution for a simply supported isotropic sandwich plate in cylindrical bending under a uniform load on the upper surface

The formulation of this problem is made in the main text, Eqs. (1)–(24). In this appendix we consider a problem with the load intensity on the lower surface equal to zero, and the load intensity on the upper surface-constant: $q_l = 0$, $q_u = \text{const}$. We also consider that the Young's moduli of the face sheets are different from that of the core ($E^{(1)} = E^{(3)} \neq E^{(2)}$), but the Poisson ratio is the same for all layers ($v^{(1)} = v^{(2)} = v^{(3)} = v$).

We will find an exact elasticity solution to this problem following a procedure suggested by Pikul (1977). This solution will satisfy the integral mitigated stress boundary conditions of Eqs. (20)–(22), not the exact stress boundary conditions at each point of the boundary.

Let us take shear strains of the layers in the form

$$\varepsilon_{xz}^{(k)} = R\big(z^2 - c^{(k)}\big)x, \tag{C.1}$$

where R and $c^{(k)}$ are the unknown constants, which are to be defined. Upon substitution of Eq. (C.1) into the constitutive equation (10), we receive

$$\sigma_{xz}^{(k)} = \frac{E^{(k)}}{1+v} R\big(z^2 - c^{(k)}\big)x. \tag{C.2}$$

Let us substitute expression (C.2) into the equilibrium equations (1) and (2), and integrate them with respect to x and z yielding

$$\sigma_{xx}^{(k)} = -\frac{E^{(k)}}{1+v} R\big[x^2 z + \varphi^{(k)}(z)\big], \tag{C.3}$$

$$\sigma_{zz}^{(k)} = -\frac{E^{(k)}}{1+v} R\left[\frac{z^3}{3} - c^{(k)} z + \psi^{(k)}(x)\right], \tag{C.4}$$

where $\varphi^{(k)}(z)$ and $\psi^{(k)}(x)$ are the arbitrary functions of integration. The substitution of expressions (C.3) and (C.4) into the constitutive equations (12) and (13) yields

$$\varepsilon_{xx}^{(k)} = -(1-v)R\left[x^2 z + \varphi^{(k)}(z) - \frac{v}{1-v}\left(\frac{z^3}{3} - c^{(k)} z + \varphi^{(k)}(z)\right)\right], \tag{C.5}$$

$$\varepsilon_{zz}^{(k)} = -(1-v)R\left[\frac{z^3}{3} - c^{(k)} z + \psi^{(k)}(x) - \frac{v}{1-v}\big(x^2 z + \varphi^{(k)}(z)\big)\right]. \tag{C.6}$$

The substitution of Eq. (C.5) into Eq. (3) and integration of the resulting equation with respect to x yields

$$u^{(k)} = -(1-v)R\left[\frac{x^3}{3} z + x\varphi^{(k)}(z) - \frac{v}{1-v}\left(\frac{z^3}{3} - c^{(k)} z\right)x - \frac{v}{1-v}\int \psi^{(k)}(x)\,\mathrm{d}x + \chi^{(k)}(z)\right], \tag{C.7}$$

where $\chi^{(k)}(z)$ is an arbitrary function of integration. The substitution of Eq. (C.6) into Eq. (4) and integration of the resulting equation with respect to z yields

$$w^{(k)} = -R(1-v)\left[\frac{z^4}{12} - c^{(k)}\frac{z^2}{2} + z\psi^{(k)}(x) - \frac{v}{1-v}x^2\frac{z^2}{2} - \frac{v}{1-v}\int \varphi^{(k)}(z)\,\mathrm{d}z + \lambda^{(k)}(x)\right]. \tag{C.8}$$

We receive the second form of expression for $\varepsilon_{xz}^{(k)}$ upon substitution of expressions (C.7) and (C.8) for displacements into the strain–displacement equation (5)

$$\varepsilon_{xz}^{(k)} = -R(1-v)\left[\frac{x^3}{3} + x\frac{\mathrm{d}\varphi^{(k)}(z)}{\mathrm{d}z} - \frac{v}{1-v}\left(z^2 - c^{(k)}\right)x + \frac{\mathrm{d}\chi^{(k)}(z)}{\mathrm{d}z} + z\frac{\mathrm{d}\psi^{(k)}(x)}{\mathrm{d}x} - \frac{v}{1-v}z^2x + \frac{\mathrm{d}\lambda^{(k)}(x)}{\mathrm{d}x}\right]. \tag{C.9}$$

An exact elasticity solution is possible if both expressions for $\varepsilon_{xz}^{(k)}$, Eqs. (C.1) and (C.9), are identically equal

$$R\left(z^2 - c^{(k)}\right)x \equiv -R(1-v)\left[\frac{x^3}{3} + x\frac{\mathrm{d}\varphi^{(k)}(z)}{\mathrm{d}z} - \frac{v}{1-v}\left(z^2 - c^{(k)}\right)x + \frac{\mathrm{d}\chi^{(k)}(z)}{\mathrm{d}z} + z\frac{\mathrm{d}\psi^{(k)}(x)}{\mathrm{d}x} - \frac{v}{1-v}z^2x + \frac{\mathrm{d}\lambda^{(k)}(x)}{\mathrm{d}x}\right]. \tag{C.10}$$

In order to find the functions $\varphi^{(k)}(z)$, $\psi^{(k)}(x)$, $\lambda^{(k)}(x)$ and $\chi^{(k)}(z)$, which make the identity (C.10) possible, let us represent the functions $\varphi^{(k)}(z)$, $\psi^{(k)}(x)$ and $\lambda^{(k)}(x)$ in the form

$$\begin{aligned}
\varphi^{(k)}(z) &= \varphi_1^{(k)}(z) + \varphi_2^{(k)}(z) + \varphi_3^{(k)}(z) + \varphi_4^{(k)}(z),\\
\psi^{(k)}(x) &= \psi_1^{(k)}(x) + \psi_2^{(k)}(x),\\
\lambda^{(k)}(x) &= \lambda_1^{(k)}(x) + \lambda_2^{(k)}(x).
\end{aligned} \tag{C.11}$$

The substitution of Eq. (C.11) into Eq. (C.10) yields

$$\left(\frac{x^3}{3} + \frac{\mathrm{d}\lambda_2^{(k)}(x)}{\mathrm{d}x}\right) + \left[x\frac{\mathrm{d}\varphi_1^{(k)}(z)}{\mathrm{d}z} - \frac{v}{1-v}\left(z^2 - c^{(k)}\right)x + \frac{\left(z^2 - c^{(k)}\right)x}{1+v}\right] + \left(x\frac{\mathrm{d}\varphi_2^{(k)}(z)}{\mathrm{d}z} - \frac{v}{v-1}xz^2\right) + \left(x\frac{\mathrm{d}\varphi_3^{(k)}(z)}{\mathrm{d}z} + z\frac{\mathrm{d}\psi_1^{(k)}(x)}{\mathrm{d}x}\right) + \left(x\frac{\mathrm{d}\varphi_4^{(k)}(z)}{\mathrm{d}z} + \frac{\mathrm{d}\lambda_1^{(k)}}{\mathrm{d}x}\right) + \left(\frac{\mathrm{d}\chi^{(k)}(z)}{\mathrm{d}z} + z\frac{\mathrm{d}\psi_2^{(k)}(x)}{\mathrm{d}x}\right) \equiv 0. \tag{C.12}$$

The identity (C.12) will take place, if each term in brackets in Eq. (C.12) is equal to zero. This leads to differential equations for the functions $\varphi_i^{(k)}(z)$ $(i = 1,2,3,4)$, $\psi_1^{(k)}(x)$, $\psi_2^{(k)}(x)$, $\lambda_1^{(k)}(x)$, $\lambda_2^{(k)}(x)$. When we solve these differential equations and substitute the found functions into expressions (C.11), we find

$$\begin{aligned}
\varphi^{(k)}(z) &= \frac{v(3-v)-1}{1-v}\left(\frac{z^3}{3} - c^{(k)}z\right) + \frac{v}{1-v}\frac{z^3}{3} + \beta^{(k)}\frac{z^2}{2} + \kappa^{(k)}z + a^{(k)},\\
\psi^{(k)}(x) &= -\beta^{(k)}\frac{x^2}{2} + e^{(k)}x + b^{(k)},\\
\lambda^{(k)}(x) &= -\frac{x^4}{12} - \ -^{(k)}\frac{x^2}{2} + d^{(k)},\\
\chi^{(k)}(z) &= -e^{(k)}\frac{z^2}{2} + \kappa^{(k)},
\end{aligned} \tag{C.13}$$

where $a^{(k)}$, $b^{(k)}$, $d^{(k)}$, $e^{(k)}$, $\beta^{(k)}$, $-^{(k)}$ and $\kappa^{(k)}$ are constants of integration. The substitution of Eq. (C.13) into Eqs. (C.3), (C.4), (C.7) and (C.8) yields

$$\sigma_{xx}^{(k)} = -\frac{E^{(k)}}{1+v}R\left[x^2z + \frac{v(3-v)-1}{1-v}\left(\frac{z^3}{3} - c^{(k)}z\right) + \frac{v}{1-v}\frac{z^3}{3} + \beta^{(k)}\frac{z^2}{2} + \ -^{(k)}z + a^{(k)}\right], \tag{C.14}$$

$$\sigma_{zz}^{(k)} = -\frac{E^{(k)}}{1+v}R\left[\frac{z^3}{3} - c^{(k)}z - \beta^{(k)}\frac{x^2}{2} + e^{(k)}x + b^{(k)}\right], \tag{C.15}$$

$$\begin{aligned} u^{(k)} = {} & -(1-v)R\left[\frac{x^3}{3}z + \frac{1}{3}(v-1)\left(z^2 - 3c^{(k)}\right)zx + \frac{v}{1-v}\frac{z^3}{3}x + \beta^{(k)}\frac{z^2}{2}x + \;-{}^{(k)}zx + a^{(k)}x \right. \\ & \left. + \frac{v}{1-v}\left(\beta^{(k)}\frac{x^3}{6} - e^{(k)}\frac{x^2}{2} - b^{(k)}x\right) - e^{(k)}\frac{z^2}{2} + \kappa^{(k)}\right], \end{aligned} \tag{C.16}$$

$$\begin{aligned} w^{(k)} = {} & -R(1-v)\left[\frac{z^4}{12} - c^{(k)}\frac{z^2}{2} + z\left(-\beta^{(k)}\frac{x^2}{2} + e^{(k)}x + b^{(k)}\right) - \frac{v}{1-v}\frac{x^2z^2}{2} - \frac{v^2(3-v)-v}{(1-v)^2}\left(\frac{z^4}{12}\right.\right. \\ & \left.\left. - c^{(k)}\frac{z^2}{2}\right) - \frac{v^2}{(1-v)^2}\frac{z^4}{12} - \frac{v}{1-v}\beta^{(k)}\frac{z^3}{6} - \frac{v}{1-v} - (k)\frac{z^2}{2} - \frac{v}{1-v}a^{(k)}z - \frac{x^4}{12} - \;-{}^{(k)}\frac{x^2}{2} + d^{(k)}\right]. \end{aligned} \tag{C.17}$$

The substitution of expressions (C.14)–(C.17) into the boundary conditions, symmetry conditions and continuity conditions of Eqs. (16)–(25) yields equations for the constants of integration. By solving these equations and substituting expressions for the constants of integration into expressions (C.2), (C.14) and (C.15) for stresses, we receive

$$\sigma_{xz}^{(1)} = \frac{q_u}{b}\frac{6E^{(1)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left(z^2 - \frac{1}{4}h^2\right)\left(x - \frac{1}{2}L\right), \tag{C.18}$$

$$\sigma_{xz}^{(2)} = \frac{q_u}{b}\frac{6E^{(2)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left(z^2 - \frac{1}{4}\frac{h^2E^{(1)} - t^2E^{(1)} + t^2E^{(2)}}{E^{(2)}}\right)\left(x - \frac{1}{2}L\right), \tag{C.19}$$

$$\sigma_{xz}^{(3)} = \frac{q_u}{b}\frac{6E^{(1)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left(z^2 - \frac{1}{4}h^2\right)\left(x - \frac{1}{2}L\right), \tag{C.20}$$

$$\begin{aligned} \sigma_{xx}^{(1)} = {} & \frac{q_u}{b}\frac{6E^{(1)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left\{(L-x)xz - \frac{1}{5}\left[\frac{3}{4}\left(h^2 + t^2\right) + ht\right]z + \frac{2}{3}z^3 - \frac{1}{15}ht(h+t)\right. \\ & \left. - \frac{1}{60}\left(t^3 + h^3\right)\right\}, \end{aligned} \tag{C.21}$$

$$\sigma_{xx}^{(2)} = \frac{q_u}{b}\frac{E^{(2)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left[6(L-x)xz + 4z^3 - \frac{3}{5}t^2z\right], \tag{C.22}$$

$$\begin{aligned} \sigma_{xx}^{(3)} = {} & \frac{q_u}{b}\frac{6E^{(1)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left\{(L-x)xz - \frac{1}{5}\left[\frac{3}{4}\left(h^2 + t^2\right) + ht\right]z + \frac{2}{3}z^3 + \frac{1}{15}ht(t+h)\right. \\ & \left. + \frac{1}{60}\left(t^3 + h^3\right)\right\}, \end{aligned} \tag{C.23}$$

$$\sigma_{zz}^{(1)} = -\frac{q_u}{b}\frac{6E^{(1)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left(\frac{1}{3}z^3 - \frac{1}{4}h^2z - \frac{1}{12}h^3\right), \tag{C.24}$$

$$\sigma_{zz}^{(2)} = -\frac{q_{\mathrm{u}}}{b}\frac{6E^{(2)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left[\frac{1}{3}z^3 - \frac{1}{4}\frac{h^2E^{(1)} - t^2E^{(1)} + t^2E^{(2)}}{E^{(2)}}z - \frac{1}{12}\frac{h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}}{E^{(2)}}\right], \tag{C.25}$$

$$\sigma_{zz}^{(3)} = -\frac{q_{\mathrm{u}}}{b}\frac{6E^{(1)}}{(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)})}\left[\frac{1}{3}z^3 - \frac{1}{4}h^2z - \frac{1}{12}h^3 + \frac{1}{6}\left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3\right], \tag{C.26}$$

$$\begin{aligned} w_0 &\equiv w|_{z=0} \\ &= -\frac{q}{b}\frac{3(1-\nu^2)}{E^{(1)}h^3 - E^{(1)}t^3 + E^{(2)}t^3}x(L-x)\left[\frac{1}{6}\left(x - \frac{L}{2}\right)^2 - \frac{5}{24}L^2 - \left(\frac{2}{5} + \frac{1}{4}\frac{\nu}{1-\nu}\right)\right. \\ &\quad \left.\times\frac{E^{(1)}h^5 - E^{(1)}t^5 + E^{(2)}t^5}{E^{(1)}h^3 - E^{(1)}t^3 + E^{(2)}t^3} - \frac{3}{4}\frac{E^{(1)}(h^2 - t^2)}{E^{(1)}h^3 - E^{(1)}t^3 + E^{(2)}t^3}\frac{E^{(1)} - E^{(2)}}{1-\nu}\left(\frac{t(h^2 - t^2)}{E_2} - \frac{\nu t^3}{3E^{(1)}}\right)\right]. \end{aligned} \tag{C.27}$$

$$\frac{\mathrm{d}^4w_0}{\mathrm{d}x^4} = -12Q\frac{\nu^2 - 1}{E^{(1)}(h^3 - t^3) + E^{(2)}t^3} \tag{C.28}$$

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